Efficient Hidden Vector Encryptions and Its Applications

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Abstract

Predicate encryption is a new paradigm of public key encryption that enables searches on encrypted data. Using the predicate encryption, we can search keywords or attributes on encrypted data without decrypting the ciphertexts. In predicate encryption, a ciphertext is associated with attributes and a token corresponds to a predicate. The token that corresponds to a predicate $f$ can decrypt the ciphertext associated with attributes $x$ if and only if $f(x) = 1$. Hidden vector encryption (HVE) is a special kind of predicate encryption. HVE supports the evaluation of conjunctive equality, comparison, and subset operations between attributes in ciphertexts and attributes in tokens. Currently, several HVE schemes were proposed where the ciphertext size, the token size, and the decryption cost are proportional to the number of attributes in the ciphertext. In this thesis, we consider the efficiency, the generality, and the security of HVE schemes. The results of this thesis are described as follows.

The first results of this thesis are efficient HVE schemes where the token consists of just four group elements and the decryption only requires four bilinear map computations, independent of the number of attributes in the ciphertext. The construction uses composite order bilinear groups and is selectively secure under the well-known assumptions. The second results are efficient HVE schemes that are secure under any kind of pairing types. To achieve our goals, we proposed a general framework that converts HVE schemes from composite order bilinear groups to prime order bilinear groups. Using the framework, we convert the previous HVE schemes from composite order bilinear groups to prime order bilinear groups. The third results are fully secure HVE schemes with short tokens. Previous HVE schemes were proven to be secure only in the selective security model where the capabilities of the adversaries are severely restricted. Using the dual system encryption techniques, we construct fully secure HVE schemes with match revealing property in composite order groups.
# Contents

1 Introduction ............................................. 4
   1.1 Overview ......................................... 4
   1.2 Our Motivations .................................. 5
   1.3 Our Contributions ............................... 6
   1.4 Related Works .................................. 6

2 Background ............................................ 8
   2.1 Hidden Vector Encryption ........................ 8
   2.2 Security Model .................................. 9
       2.2.1 Selective Security Model .................. 9
       2.2.2 Full Security Model ....................... 9
   2.3 Bilinear Groups .................................. 10
       2.3.1 Bilinear Groups of Composite Order .... 10
       2.3.2 Bilinear Groups of Prime Order .......... 10
       2.3.3 Asymmetric Bilinear Groups of Prime Order . 10
   2.4 Complexity Assumptions ........................ 11
       2.4.1 Assumptions in Composite Order Bilinear Groups .... 11
       2.4.2 Assumptions in Prime Order Bilinear Groups ... 12
       2.4.3 Assumptions in Asymmetric Bilinear Groups ... 12

3 Previous Work ....................................... 15
   3.1 Trivial Construction ............................. 15
       3.1.1 PE of Boneh and Waters ................. 15
   3.2 Extreme Generalization of AIBE ................ 16
       3.2.1 HVE of Boneh and Waters ............... 16
       3.2.2 dHVE of Shi and Waters ............... 17
   3.3 HVE from Inner Product Encryption ............. 18
       3.3.1 IPE of Katz, Sahai, and Waters ....... 19
       3.3.2 Conversion from IPE to HVE ............ 20

4 Efficient HVE with Short Tokens ...................... 21
   4.1 Overview ......................................... 21
   4.2 HVE in Composite Order Groups ................ 22
       4.2.1 Construction ................................ 22
       4.2.2 Correctness ................................ 23
4.2.3 Security

4.3 HVE in Asymmetric Bilinear Groups
4.3.1 Construction
4.3.2 Security
4.3.3 Discussion
4.3.4 Freeman Method

5 Convert HVE from Composite to Prime Order Groups
5.1 Overview
5.2 Framework
5.3 Conversion 1: BW-HVE
5.3.1 Construction
5.3.2 Correctness
5.3.3 Security
5.4 Conversion 2: LL-HVE
5.4.1 Construction
5.4.2 Correctness
5.4.3 Security
5.5 Conversion 3: SW-dHVE
5.5.1 Construction
5.5.2 Correctness
5.5.3 Security

6 Fully Secure HVE with Short Tokens
6.1 Overview
6.2 HVE in Composite Order Groups
6.2.1 Construction
6.2.2 Correctness
6.2.3 Complexity Assumptions
6.2.4 Security

7 Applications
7.1 Conjunctive Equality Queries
7.2 Conjunctive Comparison Queries
7.3 Conjunctive Range Queries
7.4 Subset Queries

8 Generic Group Model
8.1 Overview
8.2 Master Theorem
8.3 Analysis of Our Assumptions
8.3.1 P3DH Assumption

9 Conclusion
Chapter 1

Introduction

1.1 Overview

Public-key encryption is one of the most fundamental primitives in modern cryptography. In public-key encryption, a sender encrypts a message $M$ under a public key $PK$, and the receiver who has a private key $SK$ that corresponds to the public key $PK$ can only decrypt the ciphertext. This simple “all-or-nothing” semantics for decryption is sufficient for traditional secure communication systems. However, as the applications of public-key encryption come to be various, a more complex semantics for decryption is necessary to specify the set of receivers. For instance, suppose that the ciphertexts associated with keywords are in a database server, and a user who has permission to read the ciphertexts that are associated with some keywords may want to decrypt that ciphertexts. Predicate encryption provides this kind of complex semantics in public-key encryption. In predicate encryption, a ciphertext is associated with attributes and a token corresponds to a predicate. The token $TK_f$ that corresponds to a predicate $f$ can decrypt the ciphertext $CT$ that is associated with attributes $x$ if and only if $f(x) = 1$. A ciphertext in predicate encryption hides not only a message $M$ but also attributes $x$. Currently, the expressiveness of predicates in predicate encryption is limited. The most expressive predicate encryption scheme is the one proposed by Katz, Sahai, and Waters in [24], and it supports inner product predicates.

Predicate encryption enables efficient data processing in the cloud computing systems where users’ data is stored in un-trusted remote servers. In the case of traditional public-key encryption, a user encrypts messages and then uploads the ciphertexts to the remote servers. If the user needs information about the ciphertexts, then he should download all the ciphertexts from the remote servers to decrypt them. Thus, this approach demands unnecessary data transfers and data decryption. In the case of predicate encryption, a user creates ciphertexts that are associated with related attributes $x$ and then stores them in the remote servers. If the user wishes to acquire information about the ciphertexts, then he generates a token $TK_f$ that matches a predicate $f$ and transfers the token to the remote server. Next the remote server retrieves all the ciphertexts that satisfy $f(x) = 1$ using the token $TK_f$ by evaluating $f(x)$, and then it returns the retrieved ciphertexts to the user. In this case, the remote server cannot learn any information except the boolean value of $f(x)$.

Hidden vector encryption (HVE) is a particular kind of predicate encryption and it was introduced by Boneh and Waters [13]. HVE supports evaluations of conjunctive equality, comparison, and subset predicates on encrypted data. For example, if a ciphertext is associated with a vector $x = (x_1, \ldots, x_l)$ of attributes and a token is associated with a vector $\sigma = (\sigma_1, \ldots, \sigma_l)$ of attributes where an attribute is in a set $\Sigma$, then it can evaluate predicates like $(x_i = \sigma_i)$, $(x_i \geq \sigma)$, and $(x_i \in A)$ where $A$ is a subset of $\Sigma$. 
Additionally, it supports conjunctive combination of these primitive predicates by extending the size of ciphertexts. After the introduction of HVE based on composite order bilinear groups, several HVE schemes have been proposed in [23][24][29][36]. Katz, Sahai, and Waters [24] proposed a predicate encryption scheme that supports inner product predicates and they showed that it implies an HVE scheme. Shi and Waters [36] presented a delegatable HVE scheme that enables the delegation of user’s capabilities to others, and they showed that it implies an anonymous hierarchical identity-based encryption (HIBE) scheme. Iovino and Persiano [23] constructed an HVE scheme based on prime order bilinear groups, but the number of attributes in $\Sigma$ is restricted when it is compared to other HVE schemes. Okamoto and Takashima [29] proposed a hierarchical predicate encryption scheme for inner products under prime order bilinear groups, and it also implies an HVE scheme.

1.2 Our Motivations

When cryptographic schemes are applied to the real applications, they should be efficient and secure against strong adversaries. To meet this requirements, we should consider the three issues of efficiency, generality, and security.

The first issue is the efficiency. Generally efficiency is measured in terms of the size of ciphertexts and the cost of search operations. If the size of ciphertexts in previous HVE schemes is considered, the number of ciphertext elements is proportional to the number of attributes in ciphertexts, and the size of group elements of ciphertexts is proportional to the size of group order. If the cost of search operations is considered, the number of pairing operations is proportional to the number of token elements. Therefore, it is important to shorten the number of token elements to reduce the cost of search operations. Additionally, it is better to use prime order bilinear groups than to use composite order bilinear groups in terms of efficiency.

The second issue is the generality such that the HVE schemes can be based on any kind of pairing types. Pairing in bilinear groups is classified as three types [17]. Bilinear groups with Type 1 pairing corresponds to symmetric bilinear groups, and bilinear groups with Type 2 or Type 3 pairing correspond to asymmetric bilinear groups. The previous HVE schemes were constructed under composite order bilinear groups that exist in Type 1 pairing, or under asymmetric bilinear groups of prime order. The problem of cryptographic scheme’s dependency to specific pairing types is that the cryptographic scheme can be useless if a successful attack for a specific pairing type is found. The solution for this problem is to design a cryptographic scheme to be independent of a specific pairing type. That is, the cryptographic scheme should have generality that it is secure under any kind of pairing types. Thus, if a security weakness of some pairing types is found, then the security of the scheme is guaranteed by just replacing the underlying pairing type without re-designing the cryptographic scheme. However, it is an open problem to construct an efficient HVE scheme that is secure under any kind of pairing types.

The third issue is the security such that the HVE schemes should be secure against strong adversaries. The security models of HVE are categorized as two kinds: the selective security model and the full security model. In the selective security model, an adversary should commit target vectors before he receive a public key. Thus, this model severely restricts the capability of the adversary. In the full security model, the adversary commits target vectors at the challenge stage. The previous HVE schemes were proved in the selective security model since it is easy to prove its security in this model. However, the right security model for HVE is the full security model that does not restrict the ability of the adversary. Therefore, it is an important problem to design an HVE scheme that is secure in the full security model.
1.3 Our Contributions

The results of this thesis are divided as three folds. The detailed results are described as follows.

Our first results are efficient HVE scheme with short tokens. In composite order bilinear groups, we constructed HVE schemes that have the constant size of tokens and the constant cost of pairing operations, and we proved its selective security under the decisional Bilinear Diffie-Hellman (BDH) and the decisional Composite 3-Party Diffie-Hellman (C3DH) assumptions. The ciphertext consists of \( l + O(1) \) group elements, the token consists of four group elements, and the decryption requires four pairing computations. Though our construction in composite order bilinear groups is algebraically similar to the one by Shi and Waters in [36], we achieved the constant size of tokens and the constant cost of decryption, in contrast to the construction of Shi and Waters. Additionally, we converted our construction in composite order groups to asymmetric bilinear groups of prime order where isomorphisms between two groups are not efficiently computable.

Our second results are efficient HVE schemes that are secure under any kind of pairing types. To achieve our goals, we proposed a general framework that converts HVE schemes from composite order bilinear groups to prime order bilinear groups. The conversion method of this paper is similar to the conversion method of Freeman in terms of using product groups and vector orthogonality, but it has the following three differences. The first difference is that Freeman’s method is related to the subgroup decision (SGD) assumption in prime order bilinear groups, whereas our method is not related to the SGD assumption. The second difference is that Freeman’s method only works in asymmetric bilinear groups of prime order, whereas our method works in any bilinear groups of prime order. The third difference is that the cryptographic schemes from Freeman’s method use complex assumptions that depend on complex basis vectors, whereas the HVE schemes from our method use simple assumptions that are independent of basis vectors. We first convert the HVE scheme of Boneh and Waters, the delegatable HVE scheme of Shi and Waters, and the efficient HVE scheme with constant cost of pairing of Lee and Lee from composite order bilinear groups to prime order bilinear groups. Next we prove that these converted HVE schemes are selectively secure under the decisional Bilinear Diffie-Hellman (BDH) and the decisional Parallel 3-party Diffie-Hellman (P3DH) assumptions. Through these conversion, we constructed the first delegatable HVE scheme and efficient HVE scheme with constant cost of pairing in any bilinear groups of prime order.

Our third results are fully secure HVE schemes with short tokens. To construct fully secure HVE schemes, we adapt the dual system encryption technique of Waters [28, 40]. In the dual system encryption, the ciphertexts and tokens can be a normal-type or a semi-functional type. The dual system encryption technique achieves the full security model by using two properties such that the normal-type and the semi-functional type are indistinguishable and the decryption of the semi-functional ciphertext using the semi-functional token always fails. We propose a fully secure HVE scheme with short tokens in composite order bilinear groups with four primes.

1.4 Related Works

Predicate encryption in public-key encryption was presented by Boneh et al. [7]. They proposed a public-key encryption scheme with keyword search (PEKS) using Boneh and Franklin’s identity-based encryption (IBE) scheme [8, 9], and their construction corresponds to the implementation of an equality predicate. Abdalla et al. [1] proved that anonymous IBE implies predicate encryption of an equality query, and they proposed the definition of anonymous HIBE by extending anonymous IBE. Several anonymous HIBE constructions were proposed in [14, 33, 36]. A predicate encryption scheme for a comparison query was con-
structed by Boneh et al. in [11, 12], and it can be used to construct a fully collusion resistant traitor tracing scheme. By extending comparison predicates, Shi et al. [35] considered multi-dimensional range predicates on encrypted data under a weaker security model.

Research on predicate encryption was dramatically advanced by the introduction of HVE by Boneh and Waters [13]. An HVE scheme is a predicate encryption scheme of conjunctive equality, comparison, and subset predicates. After that, Shi and Waters [36] presented the definition of the delegation in predicate encryption, and they proposed a delegatable HVE scheme. Iovino and Persiano [23] constructed an HVE scheme based on prime order bilinear groups with a restricted number of attributes. Katz, Sahai, and Waters [24] proposed the most expressive predicate encryption scheme of inner product predicates, and they showed that it implies anonymous IBE, HVE, and predicate encryption for disjunctions, polynomials, CNF & DNF formulas, or threshold predicates. Okamoto and Takashima [29] constructed a hierarchical predicate encryption scheme for inner products under prime order bilinear groups using the notion of dual pairing vector spaces.

Predicate encryption in symmetric encryption was considered by Goldreich and Ostrovsky [21]. Song et al. [38] proposed an efficient scheme that supports an equality predicate. Shen, Shi, and Waters [34] introduced the formal definition of predicate privacy, and they presented a symmetric predicate encryption scheme with predicate privacy of inner product predicates using composite order bilinear groups. Blundo et al. [4] proposed a symmetric HVE scheme that provides weaker predicate privacy under prime order asymmetric bilinear groups.

Other research direction that is related with predicate encryption is functional encryption. In functional encryption, a ciphertext is associated with attributes \( x \), and a private key is associated with a function \( f \). If \( f(x) = 1 \), then a receiver who has a private key that corresponds to the function \( f \) can decrypt the ciphertext that corresponds to attributes \( x \). The main difference between predicate encryption and functional encryption is that the attribute hiding property is not provided in functional encryption, whereas the attribute hiding property was the essential one in predicate encryption. The identity-based encryption (IBE) is the most simple type of functional encryption, and it provide an equality function for an identity in ciphertexts [5, 8, 9, 19, 39]. The hierarchical IBE (HIBE) is an extension of IBE, and it provides a conjunctive equality function for a hierarchical identity in ciphertexts [5, 6, 14, 20, 28, 33, 40]. The attribute-based encryption (ABE) is also an extension of IBE, and it provides the most general function that consists of AND, OR, NOT, and threshold gates [3, 22, 27, 30, 32].
Chapter 2

Background

In this chapter, we define HVE and give the formal definition of its security model. Then we give the necessary background on bilinear groups and complexity assumptions.

2.1 Hidden Vector Encryption

Let $\Sigma$ be a finite set of attributes and let $\ast$ be a special symbol not in $\Sigma$. Define $\Sigma_* = \Sigma \cup \{\ast\}$. The star $\ast$ plays the role of a wild card or “don’t care” value. For a vector $\sigma = (\sigma_1, \ldots, \sigma_l) \in \Sigma_*^l$, we define a predicate $f_\sigma$ over $\Sigma_*^l$ as follows: For $x = (x_1, \ldots, x_l) \in \Sigma_*^l$, it set $f_\sigma(x) = 1$ if $\forall i : (\sigma_i = x_i$ or $\sigma_i = \ast)$, it set $f_\sigma(x) = 0$ otherwise.

An HVE scheme consists of four algorithms ($\text{Setup}$, $\text{GenToken}$, $\text{Encrypt}$, $\text{Query}$). Formally it is defined as:

$\text{Setup}(1^\lambda)$. The setup algorithm takes as input a security parameter $1^\lambda$. It outputs a public key $PK$ and a secret key $SK$.

$\text{GenToken}(\sigma, SK, PK)$. The token generation algorithm takes as input a vector $\sigma = (\sigma_1, \ldots, \sigma_l) \in \Sigma_*^l$ that corresponds to a predicate $f_\sigma$, the secret key $SK$ and the public key $PK$. It outputs a token $TK_\sigma$ for the vector $\sigma$.

$\text{Encrypt}(x, M, PK)$. The encrypt algorithm takes as input a vector $x = (x_1, \ldots x_l) \in \Sigma_*^l$, a message $M \in M$, and the public key $PK$. It outputs a ciphertext $CT$ for $x$ and $M$.

$\text{Query}(CT, TK_\sigma, PK)$. The query algorithm takes as input a ciphertext $CT$, a token $TK_\sigma$ for a vector $\sigma$ that corresponds to a predicate $f_\sigma$, and the public key $PK$. It outputs $M$ if $f_\sigma(x) = 1$ or outputs $\bot$ otherwise.

The scheme should satisfy the following correctness property: for all $x \in \Sigma^l, M \in M, \sigma \in \Sigma_*^l$, let $(PK, SK) \leftarrow \text{Setup}(1^\lambda), CT \leftarrow \text{Encrypt}(x, M, PK)$, and $TK_\sigma \leftarrow \text{GenToken}(\sigma, SK, PK)$.

- If $f_\sigma(x) = 1$, then $\text{Query}(CT, TK_\sigma, PK) = M$.
- If $f_\sigma(x) = 0$, then $\text{Query}(CT, TK_\sigma, PK) = \bot$ with all but negligible probability.
2.2 Security Model

2.2.1 Selective Security Model

We define the selective security model of HVE as the following game between a challenger $C$ and an adversary $A$:

**Init:** $A$ submits two vectors $x_0, x_1 \in \Sigma^l$.

**Setup:** $C$ runs the setup algorithm and keeps the secret key $SK$ to itself, then it gives the public key $PK$ to $A$.

**Query 1:** $A$ adaptively requests a polynomial number of tokens for vectors $\sigma_1, \ldots, \sigma_{q_1}$ that correspond to predicates $f_{\sigma_1}, \ldots, f_{\sigma_{q_1}}$ subject to the restriction that $f_{\sigma_i}(x_0) = f_{\sigma_i}(x_1)$ for all $i$. In responses, $C$ gives the corresponding tokens $TK_{\sigma_i}$ to $A$.

**Challenge:** $A$ submits two messages $M_0, M_1$ subject to the restriction that if there is an index $i$ such that $f_{\sigma_i}(x_0) = f_{\sigma_i}(x_1) = 1$ then $M_0 = M_1$. $C$ chooses a random coin $\gamma$ and gives a ciphertext $CT$ of $(x_\gamma, M_\gamma)$ to $A$.

**Query 2:** $A$ continues to request tokens for vectors $\sigma_{q_1+1}, \ldots, \sigma_q$ that correspond to predicates $f_{\sigma_{q_1+1}}, \ldots, f_{\sigma_q}$ subject to the two restrictions as before.

**Guess:** $A$ outputs a guess $\gamma'$. If $\gamma = \gamma'$, it outputs 0. Otherwise, it outputs 1.

The advantage of $A$ is defined as $\text{Adv}^\text{HVE}_A = \left| \Pr[\gamma = \gamma'] - \frac{1}{2} \right|$ where the probability is taken over the coin tosses made by $A$ and $C$.

**Definition 2.2.1.** We say that an HVE scheme is selectively secure if all probabilistic polynomial-time adversaries have at most a negligible advantage in the above game.

2.2.2 Full Security Model

We define the full security model of HVE as the following game between a challenger $C$ and an adversary $A$:

**Setup:** $C$ runs the setup algorithm and keeps the secret key $SK$ to itself, then it gives the public key $PK$ to $A$.

**Query 1:** $A$ adaptively requests a polynomial number of tokens for vectors $\sigma_1, \ldots, \sigma_{q_1}$ that correspond to predicates $f_{\sigma_1}, \ldots, f_{\sigma_{q_1}}$. In responses, $C$ gives the corresponding tokens $TK_{\sigma_i}$ to $A$.

**Challenge:** $A$ submits two vectors $x_0, x_1 \in \Sigma^l$ and two messages $M_0, M_1$ subject to the following two restrictions:

- For all $i \in \{1, \ldots, q_1\}$, $f_{\sigma_i}(x_0) = f_{\sigma_i}(x_1)$.
- If $\exists i \in \{1, \ldots, q_1\}$ such that $f_{\sigma_i}(x_0) = f_{\sigma_i}(x_1) = 1$, then $M_0 = M_1$.

**Query 2:** $A$ continues to request tokens for vectors $\sigma_{q_1+1}, \ldots, \sigma_q$ that correspond to predicates $f_{\sigma_{q_1+1}}, \ldots, f_{\sigma_q}$ subject to the two restrictions as before.


Guess: $A$ outputs a guess $\gamma'$. If $\gamma = \gamma'$, it outputs 0. Otherwise, it outputs 1.

The advantage of $A$ is defined as $\operatorname{Adv}^\text{HVE}_{A} = |\Pr[\gamma = \gamma'] - 1/2|$ where the probability is taken over the coin tosses made by $A$ and $C$.

**Definition 2.2.2.** We say that an HVE scheme is fully secure (with match concealing) if all probabilistic polynomial-time adversaries have at most a negligible advantage in the above game.

**Definition 2.2.3.** We say that an HVE scheme is fully secure (with match revealing) if all probabilistic polynomial-time adversaries have at most a negligible advantage in the above game with restriction that the adversary can not query predicates such that $f_{\sigma}(x_0) = f_{\sigma}(x_1) = 1$.

### 2.3 Bilinear Groups

#### 2.3.1 Bilinear Groups of Composite Order

The composite order bilinear groups were first introduced in [10]. Let $n = pqr$ where $p$, $q$, and $r$ are distinct prime numbers. Let $G$ and $G_T$ be two multiplicative cyclic groups of composite order $n$ and $g$ be a generator of $G$. The bilinear map $e : G \times G \rightarrow G_T$ has the following properties:

1. **Bilinearity:** $\forall u, v \in G$ and $\forall a, b \in \mathbb{Z}_n$, $e(u^a, v^b) = e(u, v)^{ab}$.

2. **Non-degeneracy:** $\exists g$ such that $e(g, g) \neq 1$, that is, $e(g, g)$ is a generator of $G_T$.

We say that $G$ is a bilinear group if the group operations in $G$ and $G_T$ as well as the bilinear map $e$ are all efficiently computable. Furthermore, we assume that the description of $G$ and $G_T$ includes generators of $G$ and $G_T$ respectively.

We use the notation $G_p, G_q, G_r$ to denote the subgroups of order $p, q, r$ of $G$ respectively. Similarly, we use the notation $G_{T,p}, G_{T,q}, G_{T,r}$ to denote the subgroups of order $p, q, r$ of $G_T$ respectively.

#### 2.3.2 Bilinear Groups of Prime Order

Let $G$ and $G_T$ be multiplicative cyclic groups of prime $p$ order. Let $g$ be a generator of $G$. The bilinear map $e : G \times G \rightarrow G_T$ has the following properties:

1. **Bilinearity:** $\forall u, v \in G$ and $\forall a, b \in \mathbb{Z}_p$, $e(u^a, v^b) = e(u, v)^{ab}$.

2. **Non-degeneracy:** $\exists g$ such that $e(g, g)$ has order $p$, that is, $e(g, g)$ is a generator of $G_T$.

We say that $(p, G, G_T, e)$ are bilinear groups if the group operations in $G$ and $G_T$ as well as the bilinear map $e$ are all efficiently computable.

#### 2.3.3 Asymmetric Bilinear Groups of Prime Order

Let $G, \hat{G}$, and $G_T$ be multiplicative cyclic groups of prime $p$ order where $G \neq \hat{G}$. Let $g, \hat{g}$ be generators of $G, \hat{G}$, respectively. The asymmetric bilinear map $e : G \times \hat{G} \rightarrow G_T$ has the following properties:

1. **Bilinearity:** $\forall u \in G, \forall v \in \hat{G}$ and $\forall a, b \in \mathbb{Z}_p$, $e(u^a, \hat{g}^b) = e(u, \hat{g})^{ab}$.

2. **Non-degeneracy:** $\exists g, \hat{g}$ such that $e(g, \hat{g}) \neq 1$, that is, $e(g, \hat{g})$ is a generator of $G_T$.

We say that $G, \hat{G}, G_T$ are asymmetric bilinear groups with no efficiently computable isomorphisms if the group operations in $G, \hat{G}$ and $G_T$ as well as the bilinear map $e$ are all efficiently computable, but there are no efficiently computable isomorphisms between $G$ and $\hat{G}$.  

10
2.4 Complexity Assumptions

2.4.1 Assumptions in Composite Order Bilinear Groups

We introduce three assumptions under composite order bilinear groups. The decisional composite bilinear Diffie-Hellman (cBDH) assumption was used to construct an HVE scheme in [13]. It is a natural extension of the decisional BDH assumption in [8] from prime order bilinear groups to composite order bilinear groups. The bilinear subgroup decision (BSD) assumption was introduced in [11] to construct a traitor tracing scheme. The decisional composite 3-party Diffie-Hellman (C3DH) assumption was used to construct an HVE scheme in [13].

Decisional composite Bilinear Diffie-Hellman (cBDH) Assumption Let \((n, \mathbb{G}, \mathbb{G}_T, e)\) be a description of the bilinear group of composite order \(n = pqr\). Let \(g_p, g_q, g_r\) be generators of subgroups of order \(p, q, r\) of \(\mathbb{G}\) respectively. The decisional cBDH problem is stated as follows: given a challenge tuple

\[
D = ((p, q, r, \mathbb{G}, \mathbb{G}_T, e), g_p, g_q, g_r, g_p^a, g_p^b, g_p^c) \quad \text{and} \quad T,
\]

decides whether \(T = e(g_p, g_p)^{abc}\) or \(T = R\) with random choices of \(a, b, c \in \mathbb{Z}_p, R \in \mathbb{G}_T\). The advantage of \(A\) in solving the decisional cBDH problem is defined as

\[
\text{Adv}^\text{cBDH}_{A} = \left| \Pr[A(D, T = e(g_p, g_p)^{abc}) = 1] - \Pr[A(D, T = R) = 1] \right|
\]

where the probability is taken over the random choices of \(D, T\) and the random bits used by \(A\).

Definition 2.4.1. We say that the decisional cBDH assumption holds if no probabilistic polynomial-time algorithm has a non-negligible advantage in solving the decisional cBDH problem.

Bilinear Subgroup Decision (BSD) Assumption Let \((n, \mathbb{G}, \mathbb{G}_T, e)\) be a description of the bilinear group of composite order \(n = pqr\). Let \(g_p, g_q, g_r\) be generators of subgroups of order \(p, q, r\) of \(\mathbb{G}\) respectively. The BSD problem is stated as follows: given a challenge tuple

\[
D = ((n, \mathbb{G}, \mathbb{G}_T, e), g_p, g_q, g_r) \quad \text{and} \quad T,
\]

decides whether \(T = Q \in \mathbb{G}_T\) or \(T = R \in \mathbb{G}_T\) with random choices of \(Q \in \mathbb{G}_T, R \in \mathbb{G}_T\). The advantage of \(A\) in solving the BSD problem is defined as

\[
\text{Adv}^\text{BSD}_{A} = \left| \Pr[A(D, T = Q) = 1] - \Pr[A(D, T = R) = 1] \right|
\]

where the probability is taken over the random choices of \(D, T\) and the random bits used by \(A\).

Definition 2.4.2. We say that the BSD assumption holds if no probabilistic polynomial-time algorithm has a non-negligible advantage in solving the BSD problem.

Decisional Composite 3-party Diffie-Hellman (C3DH) Assumption Let \((n, \mathbb{G}, \mathbb{G}_T, e)\) be a description of the bilinear group of composite order \(n = pqr\). Let \(g_p, g_q, g_r\) be generators of subgroups of order \(p, q, r\) of \(\mathbb{G}\) respectively. The decisional C3DH problem is stated as follows: given a challenge tuple

\[
D = ((n, \mathbb{G}, \mathbb{G}_T, e), g_p, g_q, g_r, g_p^{a_1}, g_p^{a_2}, g_p^{a_3}) \quad \text{and} \quad T,
\]

decides whether \(T = g_p^{a_1}R_3\) or \(T = R\) with random choices of \(R_1, R_2, R_3 \in \mathbb{G}_q, R \in \mathbb{G}_{pq}\). The advantage of \(A\) in solving the decisional C3DH problem is defined as

\[
\text{Adv}^\text{C3DH}_{A} = \left| \Pr[A(D, T = g_p^{a_1}R_3) = 1] - \Pr[A(D, T = R) = 1] \right|
\]

where the probability is taken over the random choices for \(D, T\) and the random bits used by \(A\).
Definition 2.4.3. We say that the decisional C3DH assumption holds if no probabilistic polynomial-time algorithm has a non-negligible advantage in solving the decisional C3DH problem.

2.4.2 Assumptions in Prime Order Bilinear Groups

We introduce two assumptions under prime order bilinear groups. The decisional Bilinear Diffie-Hellman (BDH) assumption is well-known one and introduced in [8]. The decisional Parallel 3-party Diffie-Hellman (P3DH) assumption is newly introduced in this paper, and its security in generic group model is given in chapter 8.

Bilinear Diffie-Hellman (BDH) Assumption Let \((p, \mathbb{G}, \mathbb{G}_T, e)\) be a description of the bilinear group of prime order \(p\). The decisional BDH problem is stated as follows: given a challenge tuple

\[ D = ((p, \mathbb{G}, \mathbb{G}_T, e), g, g^a, g^b, g^c) \text{ and } T, \]

decides whether \( T = T_0 = e(g, g)^{abc} \) or \( T = T_1 = e(g, g)^d \) with random choices of \( a, b, c, d \in \mathbb{Z}_p \). The advantage of \( A \) in solving the decisional BDH problem is defined as

\[ \text{Adv}^{BDH}_A = \left| \Pr[A(D, T_0) = 1] - \Pr[A(D, T_1) = 1] \right| \]

where the probability is taken over the random choices of \( D, T \) and the random bits used by \( A \). We say that the decisional BDH assumption holds if no probabilistic polynomial-time algorithm has a non-negligible advantage in solving the decisional BDH problem.

Parallel 3-party Diffie-Hellman (P3DH) Assumption Let \((p, \mathbb{G}, \mathbb{G}_T, e)\) be a description of the bilinear group of prime order \(p\). The decisional P3DH problem is stated as follows: given a challenge tuple

\[ D = ((p, \mathbb{G}, \mathbb{G}_T, e), (g, f), (g^a, f^a), (g^b, f^b), (g^{ab}f^{c_1}, g^{z_1}), (g^{ab}f^{c_2}, g^{z_2})) \text{ and } T, \]

decides whether \( T = T_0 = (g^{c}f^{c_1}, g^{z_1}) \) or \( T = T_1 = (g^{d}f^{c_2}, g^{z_2}) \) with random choices of \( a, b, c, d \in \mathbb{Z}_p \) and \( z_1, z_2, z_3 \in \mathbb{Z}_p \). The advantage of \( A \) in solving the decisional P3DH problem is defined as

\[ \text{Adv}^{P3DH}_A = \left| \Pr[A(D, T_0) = 1] - \Pr[A(D, T_1) = 1] \right| \]

where the probability is taken over the random choices of \( D, T_0, T_1 \) and the random bits used by \( A \). We say that the decisional P3DH assumption holds if no probabilistic polynomial-time algorithm has a non-negligible advantage in solving the decisional P3DH problem.

Remark 1. The decisional P3DH problem can be modified as follows: an adversary is given a challenge tuple \( D = ((p, \mathbb{G}, \mathbb{G}_T, e), (g, f), (g^a, f^a), (g^b, f^b), (g^{ab}f^{c_1}, g^{z_1}), (g^{c}f^{c_2}, g^{z_2})) \) and \( T \), it decides whether \( T = T_0 = (g^{ab}f^{c_2}, g^{z_2}) \) or \( T = T_1 = (g^{d}f^{c_2}, g^{z_2}) \). However, this modified one is the same as the original one by changing the position of the challenge tuple as \( D = ((p, \mathbb{G}, \mathbb{G}_T, e), (g, f), (g^a, f^a), (g^b, f^b), (g^{ab}f^{c_1}, g^{z_1}), T) \) and \( T' = (g^{c}f^{c_2}, g^{z_2}) \). Thus, we will use any one of challenge tuple forms for the decisional P3DH assumption.

2.4.3 Assumptions in Asymmetric Bilinear Groups

We introduce three cryptographic assumptions that are secure under asymmetric bilinear groups of prime order where there are no efficiently computable isomorphisms between two groups \( \mathbb{G} \) and \( \mathbb{G} \). The decisional
asymmetric bilinear Diffie-Hellman (aBDH) assumption is the same as the decisional cBDH assumption except that it uses asymmetric bilinear groups. The decisional asymmetric Diffie-Hellman (aDH) assumption says that the traditional decisional DH assumption holds if no efficiently computable isomorphism from \( \mathbb{G} \) to \( \hat{\mathbb{G}} \) exists. The decisional asymmetric 3-party Diffie-Hellman (a3DH) assumption is an asymmetric version of the decisional C3DH assumption.

**Decisional asymmetric Bilinear Diffie-Hellman (aBDH) Assumption** Let \( (p, \mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T, e) \) be a description of the asymmetric bilinear group of prime order \( p \) with no efficiently computable isomorphism from \( \mathbb{G} \) to \( \hat{\mathbb{G}} \). The decisional aBDH problem is stated as follows: given a challenge tuple

\[
D = ((p, \mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T, e), g, g^a, g^b, g^c, \hat{g}, \hat{g}^a, \hat{g}^b) \text{ and } T,
\]

decides whether \( T = e(g, \hat{g})^{abc} \) or \( T = R \) with random choices of \( a, b, c \in \mathbb{Z}_p, R \in \mathbb{G}_T \). The advantage of \( A \) in solving the decisional aBDH problem is defined as

\[
\text{Adv}_{A}^{\text{aBDH}} = \left| \Pr[A(D, T = e(g, \hat{g})^{abc}) = 1] - \Pr[A(D, T = R) = 1] \right|
\]

where the probability is taken over the random choices of \( D, T \) and the random bits used by \( A \).

**Definition 2.4.4.** We say that the decisional aBDH assumption holds if no probabilistic polynomial-time algorithm has a non-negligible advantage in solving the decisional aBDH problem.

**Decisional asymmetric Diffie-Hellman (aDH) Assumption** Let \( (p, \mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T, e) \) be a description of the asymmetric bilinear group of prime order \( p \) with no efficiently computable isomorphisms between \( \mathbb{G} \) and \( \hat{\mathbb{G}} \). Let \( g, \hat{g} \) be generators of \( \mathbb{G}, \hat{\mathbb{G}} \) respectively. The decisional aDH problem is stated as follows: given a challenge tuple

\[
D = ((p, \mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T, e), g, \hat{g}, \hat{g}^a, \hat{g}^b) \text{ and } T,
\]

decides whether \( T = \hat{g}^{ab} \) or \( T = R \) with random choices of \( a, b \in \mathbb{Z}_p, R \in \hat{\mathbb{G}} \). The advantage of \( A \) in solving the decisional aDH problem is defined as

\[
\text{Adv}_{A}^{\text{aDH}} = \left| \Pr[A(D, T = \hat{g}^{ab}) = 1] - \Pr[A(D, T = R) = 1] \right|
\]

where the probability is taken over the random choices of \( D, T \) and the random bits used by \( A \).

**Definition 2.4.5.** We say that the decisional aDH assumption holds if no probabilistic polynomial-time algorithm has a non-negligible advantage in solving the decisional aDH problem.

**Decisional asymmetric 3-party Diffie-Hellman (a3DH) Assumption** Let \( (p, \mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T, e) \) be a description of the asymmetric bilinear group of prime order \( p \) with no efficiently computable isomorphism from \( \mathbb{G} \) to \( \hat{\mathbb{G}} \). Let \( g, \hat{g} \) be generators of \( \mathbb{G}, \hat{\mathbb{G}} \) respectively. The decisional a3DH is stated as follows: given a challenge tuple

\[
D = ((p, \mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T, e), g, g^a, g^b, g^{abc}, \hat{g}, \hat{g}^a, \hat{g}^b) \text{ and } T,
\]

decides whether \( T = g^c \) or \( T = R \) with random choice of \( a, b, c \in \mathbb{Z}_p, R \in \mathbb{G} \). The advantage of \( A \) in solving the decisional a3DH problem is defined as

\[
\text{Adv}_{A}^{\text{a3DH}} = \left| \Pr[A(D, T = g^c) = 1] - \Pr[A(D, T = R) = 1] \right|
\]

where the probability is taken over the random choices for \( D, T \) and the random bits used by \( A \).
**Definition 2.4.6.** We say that the decisional a3DH assumption holds if no probabilistic polynomial-time algorithm has a non-negligible advantage in solving the decisional a3DH problem.

**Remark 2.** The decisional aDH assumption is equivalent to the external Diffie-Hellman (XDH) assumption. In this paper, we will use aDH instead of XDH for notational consistency.
Chapter 3

Previous Work

In this chapter, we review the previous work of HVE. After the first construction of HVE by Boneh and Waters, various HVE schemes were proposed in [13, 15, 26, 36]. The design techniques of previous HVE schemes are classified as the following three categories.

3.1 Trivial Construction

The first category is a trivial construction from public key encryption (PKE) [7, 13, 23, 25]. This method was introduced by Boneh et al. to construct a public key encryption scheme with keyword search (PEKS) using trapdoor permutations [7]. After that, Boneh and Waters showed that a searchable public key encryption for general predicates can be constructed by this method [13]. Recently, Katz and Yerukhimovich showed that it is possible to construct predicate encryption scheme from a CPA-secure PKE scheme if the number of predicate such that $f(x) = 0$ is less than a polynomial value of a security parameter [25]. The main idea of this method is to use a multiple instances of key-private PKE that was introduced by Bellare et al. [2]. The public key of searchable public key encryption consists of the public keys of key-private PKE, and each instance of public keys is mapped to each predicate. However, this method has a serious problem such that the total number of predicates is limited to the polynomial value of a security parameter.

3.1.1 PE of Boneh and Waters

Let $\Sigma$ be a finite set of binary strings. A predicate $f$ over $\Sigma$ is a function $f : \Sigma \rightarrow \{0,1\}$. We say that $x \in \Sigma$ satisfies the predicate if $f(x) = 1$. Let $\Phi$ be the set of predicates, that is, $\Phi = \{f_1, f_2, \ldots, f_m\}$. The trivial predicate encryption for any set of predicates $\Phi$ using public key encryption $\mathcal{E} = (\text{Setup}_\mathcal{E}, \text{Encrypt}_\mathcal{E}, \text{Decrypt}_\mathcal{E})$ is described as follows.

\text{Setup}(1^\lambda): The setup algorithm first run $\text{Setup}_\mathcal{E}$ $m$ times to obtain $(PK_1, SK_1), \ldots, (PK_m, SK_m)$. Then it keeps $(SK_1, \ldots, SK_m)$ as a private key and publishes a public key as $PK = (PK_1, \ldots, PK_m)$.

\text{GenToken}(j, SK, PK): The token generation algorithm takes as input an index $j$ of $P$ in $\Phi$ and the secret key $SK$. It outputs a token as $TK_j = (j, SK_j)$

\text{Encrypt}(x, M, PK): The encrypt algorithm takes as input a string $x \in \Sigma$, a message $M$ and the public key $PK$. For $i = 1, \ldots, m$, it sets $C_j \leftarrow \text{Encrypt}_\mathcal{E}(PK_j, M)$ if $f_j(x) = 1$, or it sets $C_j \leftarrow \text{Encrypt}_\mathcal{E}(PK_j, \bot)$ otherwise. It outputs a ciphertext as $CT = (C_1, \ldots, C_m)$. 
Query(CT, TK_p, PK): The query algorithm takes as input CT = (C_1, ..., C_m) and TK_j = (j, SK_j). It outputs Decrypt_E(SK_j, C_j).

**Theorem 3.1.1.** The predicate encryption scheme of Boneh and Waters is secure if E is a semantically secure public key encryption and again chosen plaintext attacks.

### 3.2 Extreme Generalization of AIBE

The second category is the extreme generalization of anonymous identity-based encryption (AIBE) [13][15][26][36]. This method was introduced by Boneh and Waters to construct an HVE scheme [13]. They used the identity-based encryption (IBE) scheme of Boneh and Boyen [5] and composite order bilinear groups to provide the anonymity of ciphertexts. After that, Shi and Waters constructed a delegatable HVE (dHVE) scheme [36]. In composite order bilinear groups, the random blinding property using subgroups provides the successful decryption. However, it is inefficient to use composite order bilinear groups since the group order of composite order bilinear groups should be larger than 1024 bits to defeat the integer factorization attacks. One way to overcome this problem of inefficiency is to use prime order bilinear groups. Freeman presented a general framework that converts cryptographic schemes from composite order bilinear groups to prime order bilinear groups [16]. Ducas also showed that HVE schemes of composite order bilinear groups are easily converted to prime order bilinear groups [15]. However, these conversion methods have a problem such that they work under asymmetric bilinear groups that are particular kinds of prime order bilinear groups [17].

#### 3.2.1 HVE of Boneh and Waters

Let Σ = Z_2^m for some integer m and set Σ_σ = Z_m ∪ {∗}. The HVE scheme of Boneh and Waters is described as follows.

**Setup(1^λ):** The setup algorithm first generates the bilinear group G of composite order n = pq where p and q are random primes of bit size Θ(λ) and p, q > m. Next, it chooses random elements v ∈ G_p, (u_1, h_1, w_1), ..., (u_l, h_l, w_l) ∈ G_p^2, and a random exponent α ∈ Z_p. It keeps these as a secret key SK. Then it chooses random elements R_v ∈ G_q and (R_{a,1}, R_{a,2}, R_{h,1}, R_{h,2}) ∈ G_q^2, and it publishes a public key PK with the description of the bilinear group G as follows

\[ PK = \left( V = vR_v, \left\{ (U_i = u_iR_{a,i}, H_i = h_iR_{h,i}, W_i = w_iR_{w,i}) \right\}_{i=1}^l, g_q, \Omega = e(v, g)^{\alpha} \right). \]

**GenToken(σ, SK, PK):** The token generation algorithm takes as input a vector σ = (σ_1, ..., σ_l) ∈ Σ^l_σ and the secret key SK. It first selects random exponents \{r_i, 1, r_i, 2\}_{i=1}^l ∈ Z_p. Let S be the set of indexes that are not wild card positions in the vector σ. Then it outputs a token as

\[ TK_σ = \left( K_0 = g^{\alpha(\prod_{i \in S} u_i^{\sigma_i} h_i)^{r_i,1} w_i^{r_i,2}}, \{ K_{i,1} = v^{r_{i,1}}, K_{i,2} = v^{r_{i,2}} \}_{i \in S} \right). \]

**Encrypt(x, M, PK):** The encrypt algorithm takes as input a vector x = (x_1, ..., x_l) ∈ Σ^l, a message \( M \in \mathcal{M} \subseteq \mathbb{G}_T \) and the public key PK. It first chooses a random exponent t ∈ Z_q and random elements Z_0, (Z_{1,1}, Z_{1,2}), ..., (Z_{l,1}, Z_{l,2}) ∈ G_q by raising \( g_q \) to random elements from \( Z_q \). Next, it outputs a ciphertext as

\[ CT = \left( C = \Omega^t M, C_0 = V^t Z_0, \left\{ C_{i,1} = (U_i^\sigma H_i)^t Z_{i,1}, C_{i,2} = W_i^t Z_{i,2} \right\}_{i=1}^l \right). \]
Query(CT, TK_σ, PK): The query algorithm takes as input a ciphertext CT and a token TK_σ of a vector σ. It first computes
\[ M \leftarrow C \cdot e(C_0, K_0)^{-1} \prod_{i \in S} e(C_{i,1}, K_{i,1}) e(C_{i,2}, K_{i,2}). \]
If \( M \notin \mathcal{M} \), it outputs ⊥ indicating that the predicate \( f_σ \) is not satisfied. Otherwise, it outputs \( M \) indicating that the predicate \( f_σ \) is satisfied.

**Theorem 3.2.1 ([13]).** The HVE scheme of Boneh and Waters is selectively secure under the decisional cBDH assumption, the BSD assumption, and the decisional C3DH assumption.

### 3.2.2 dHVE of Shi and Waters

Let \( \Sigma = \mathbb{Z}_m \) for some integer \( m \) and set \( \Sigma_σ = \mathbb{Z}_m \cup \{ \ast \} \). The dHVE scheme of Shi and Waters is described as follows.

**Setup(1^λ):** The setup algorithm first generates the bilinear group \( G \) of composite order \( n = pqr \) where \( p, q \) and \( r \) are random primes of bit size \( \Theta(λ) \) and \( p, q, r > m \). Next, it chooses random elements \( v, w_1, w_2 \in \mathbb{G}_p \), \( \{ u_i, h_i \} \in \mathbb{G}_p^2 \), and a random exponent \( w \in \mathbb{Z}_p \). It keeps these as a secret key SK. Then it chooses random elements \( R_v, R_{w,1}, R_{w,2}, \{ R_{r,i} \} \in \mathbb{G}_r \), and it publishes a public key PK with the description of the bilinear group \( G \) as follows

\[ \text{PK} = \left( V = vR_v, W_1 = w_1R_{w,1}, W_2 = w_2R_{w,2}, \{ (U_j = u_jR_{n,j}, H_j = h_jR_{r,j}) \}_{j = 1}^l, g_q, g_r, \Omega = e(v, g)^{\alpha} \right). \]

**GenToken(σ, SK, PK):** The token generation algorithm takes as input a vector \( σ = (σ_1, \ldots, σ_l) \in \Sigma_σ^l \) and the secret key SK. It first selects random exponents \( r_1, r_2, \{ r_{3,i} \} \in \mathbb{Z}_p \) and random elements \( Y_0, Y_1, Y_2, \{ Y_{3,i} \} \in \mathbb{G}_r \) by raising \( g_r \) to random exponents in \( \mathbb{Z}_m \). Let \( S \) be the set of indexes that are not wild card positions in the vector \( σ \). Then it outputs a token as

\[ K_0 = g^{α} w_1^{-r_1} w_2^{-r_2} \prod_{i \in S} (u_i^{r_{3,i}} h_i^{r_{3,i}})^Y_0, K_1 = v^{r_1} Y_1, K_2 = v^{r_2} Y_2, K_3 = v^{r_{3,i}} Y_{3,i} \}_{i \in S}. \]

Let \( S \) be the set of indexes that are delegatable fields. It selects random exponents \( \{ s_{i,j} \} \in \mathbb{Z}_p \) and random values \( \{ Y_{0,i,u}, Y_{0,i,h}, Y_{1,i,j}, Y_{2,i,j}, Y_{3,i,j} \} \in \mathbb{G}_r \). Next, it computes delegation components as

\[ \forall j \in S : L_{0,j,u} = u_1^{s_{1,j}} Y_{j,u}, L_{0,j,h} = w_1^{s_{1,j}} w_2^{s_{2,j}} \prod_{i \in S} (u_i^{s_{3,i,j}} h_i^{s_{3,i,j}}) Y_{j,h}, L_{1,j} = v^{s_{1,j}} Y_{1,j}, L_{2,j} = v^{s_{2,j}} Y_{2,j}, L_{3,j,i} = v^{s_{3,i,j}} Y_{3,i,j} \}_{i \in S, j \in \{ j \}.} \]

Finally, it outputs a token as

\[ \text{TK}_σ = \left( K_0, K_1, K_2, \{ K_{3,i} \}_{i \in S}, \{ L_{0,j,u}, L_{0,j,h}, L_{1,j}, L_{2,j}, L_{3,j,i} \}_{i \in S, j \in \{ j \}.} \right). \]

**Delegate(σ', TK_σ, PK):** The delegation algorithm takes as input an attribute vector \( σ' = (σ_1, \ldots, σ_l) \in \Sigma_σ^l \) and a token TK_σ. Without loss of generality, we assume that \( σ' \) fixes only one delegatable field of \( σ \). It is clear that we can perform delegation on multiple fields if we have an algorithm to perform delegation on one field. Suppose \( σ' \) fixes the \( k \)-th index of \( σ \). If the \( k \)-th index of \( σ' \) is set to \( \ast \), that is,
a wild-card field, then it can perform delegation by simply removing the delegation components that correspond to \( k \)-th index. Otherwise, that is, if the \( k \)-th index of \( \sigma' \) is set to some value in \( \Sigma \), then it perform delegation as follows.

Let \( S \) be the set of indexes that are not delegatable fields and wild-card fields in the vector \( \sigma' \). Note that \( k \in S \). It selects random exponents \( \mu \in \mathbb{Z}_n \) and random elements \( Y_0, Y_1, Y_2, \{ Y_{3,i} \} \in \mathbb{G}_r \) and updates the token as

\[
\begin{align*}
\tilde{K}_0 &= K_0(L_{0,k,h}^\sigma L_{0,k,h})^\mu Y_0, \\
\tilde{K}_1 &= K_1 L_{1,k}^\mu Y_1, \\
\tilde{K}_2 &= K_2 L_{2,k}^\mu Y_2, \\
\tilde{K}_{3,k} &= L_{3,k}^\mu Y_{3,k}, \quad \text{for } \{ \tilde{K}_{3,i} = K_{3,i} L_{3,k}^\mu Y_{3,i} \} \in S \setminus \{k\}.
\end{align*}
\]

Let \( S_j \) be the set of indexes that are delegatable fields in the vector \( \sigma' \). It selects random exponents \( \{ \tau_j \} \in \mathbb{Z}_n \) and random elements \( \{ Y_{0,j,u}, Y_{0,j,h}, Y_{1,j}, Y_{2,j}, \{ Y_{3,j,i} \} \in S \cup \{j\} \} \in S_r \in \mathbb{G}_r \) and re-randomize the delegation components of the token as

\[
\forall j \in S_j : L_{0,j,u} = L_{0,j,u}^\mu Y_{0,j,u}, \\
L_{0,j,h} = L_{0,j,h}^\mu (L_{0,k,u}^\sigma L_{0,k,h})^\tau_j Y_{0,j,h}, \\
L_{1,j} = L_{1,j}^\mu L_{1,j}^\tau_j Y_{1,j}, \\
L_{2,j} = L_{2,j}^\mu L_{2,j}^\tau_j Y_{2,j}, \\
L_{3,j} = L_{3,j}^\mu L_{3,j}^\tau_j Y_{3,j}, \quad \text{for } \{ \tilde{L}_{3,j,i} = L_{3,j,k}^\mu L_{3,j,k}^\tau_j Y_{3,j,i} \} \in S \setminus \{k\}.
\]

Finally, it outputs a token as

\[
\text{TK}_{\sigma'} = \left( \tilde{K}_0, \tilde{K}_1, \tilde{K}_2, \{ \tilde{K}_{3,i} \}_{i \in S}, \{ \tilde{L}_{0,j,u}, \tilde{L}_{0,j,h}, \tilde{L}_{1,j}, \tilde{L}_{2,j}, \{ \tilde{L}_{3,j,i} \}_{i \in S \cup \{j\}} \}_{j \in S_r} \right).
\]

**Encrypt(x,M,PK):** The encrypt algorithm takes as input a vector \( x = (x_1, \ldots, x_l) \in \Sigma^l \), a message \( M \in M \subseteq \mathbb{G}_T \) and the public key \( \text{PK} \). It first chooses a random exponent \( t \in \mathbb{Z}_n \) and random elements \( Z_0, Z_1, Z_2, \{ Z_{3,i} \} \in \mathbb{G}_q \). Next, it outputs a ciphertext as

\[
\text{CT} = \left( C = \Omega^t M, C_0 = V^t Z_0, C_1 = W^t_{1} Z_1, C_2 = W^t_{2} Z_2, \{ C_{3,i} = (U^t_{i} H_i)^t Z_{3,i} \}_{i=1}^l \right).
\]

**Query(CT,TK,PK):** The query algorithm takes as input a ciphertext \( \text{CT} \) and a token \( \text{TK}_{\sigma} \) of a vector \( \sigma \).

It first computes

\[
M \leftarrow C \cdot e(C_0, K_0)^{-1} \cdot e(C_1, K_1) \cdot e(C_2, K_2) \cdot \prod_{i \in S} e(C_{3,i}, K_{3,i}).
\]

If \( M \notin M \), it outputs \( \perp \) indicating that the predicate \( f_{\sigma} \) is not satisfied. Otherwise, it outputs \( M \) indicating that the predicate \( f_{\sigma} \) is satisfied.

**Theorem 3.2.2** ([36]). The delegatable HVE scheme of Shi and Waters is selectively secure under the decisional cBDH assumption, the BSD assumption, and the decisional C3DH assumption.

### 3.3 HVE from Inner Product Encryption

The third category is to use inner-product encryption (IPE) [24,29,31]. IPE is a kind of predicate encryption and it enable the evaluation of inner-product predicates between the vector of ciphertexts and the vector of tokens. Katz et al. constructed the first IPE scheme under composite order bilinear groups [24]. After...
that Okamoto and Takashima constructed a hierarchical IPE scheme using dual pairing vector spaces \[29\]. Recently, Park proposed an IPE scheme under prime order bilinear groups and proved its security under the well-known assumptions \[31\]. The main idea of converting an IPE scheme to an HVE scheme is to construct a predicate of conjunctive equality using a predicate of inner product \[24\]. Though, the expressiveness of IPE enables the evaluations of predicates like conjunctive, disjunctive, polynomials, and CNF/DNF formula, IPE has a weakness such that the number of pairing operations is linearly dependent on the number of attributes in the ciphertexts.

### 3.3.1 IPE of Katz, Sahai, and Waters

Let \( \Sigma = \mathbb{Z}_m \) for some integer \( m \) and set \( \Sigma_* = \mathbb{Z}_m \cup \{+\} \). The IPE scheme of Katz, Sahai, and Waters is described as follows.

**Setup(1\(^\lambda\))**: The setup algorithm first generates the bilinear group \( \mathbb{G} \) of composite order \( n = pqr \) where \( p, q \) and \( r \) are random primes of bit size \( \Theta(\lambda) \) and \( p, q, r > m \). Let \( g_p, g_q, \) and \( g_r \) be generators of \( \mathbb{G}_p, \mathbb{G}_q, \) and \( \mathbb{G}_r \) respectively. It chooses random elements \( \{h_{1,i}, h_{2,i}\}_{i=1}^{l}, h \in \mathbb{G}_p \) and a random exponent \( \gamma \in \mathbb{Z}_p \). It keeps these as a secret key \( SK \). Then it chooses random elements \( \{R_{1,i}, R_{2,i}\}_{i=1}^{l}, Q_{i} \in \mathbb{G}_{r} \), and it publishes a public key \( PK \) with the description of the bilinear group \( \mathbb{G} \) as follows

\[
PK = \left( g = g_{p}, Q = g_{q}R_{q}, g_{r}, \{ (H_{1,i} = h_{1,i}R_{h,1,i}, H_{2,i} = h_{2,i}R_{h,2,i}) \}_{i=1}^{l}, \Omega = e(g, h)^\gamma \right).
\]

**GenToken(\(\sigma, SK, PK\))**: The token generation algorithm takes as input a vector \( \sigma = (\sigma_1, \ldots, \sigma_l) \in \Sigma_*^l \) and the secret key \( SK \). It first selects random exponents \( \{r_{1,i}, r_{2,i}\}_{i=1}^{l} \in \mathbb{Z}_p \), random elements \( R_5 \in \mathbb{G}_r, Q_6 \in \mathbb{G}_q \), and random exponents \( f_1, f_2 \in \mathbb{Z}_q \). Then it outputs a token as

\[
TK_{\sigma} = \left( K_0 = h^7R_5Q_6 \prod_{i=1}^{l}(h_{1,i}^{r_{1,i}}, h_{2,i}^{r_{2,i}}), \{ K_{1,i} = g^{r_{1,i}}H_{1,i}^{f_1}\sigma, K_{2,i} = g^{r_{2,i}}H_{2,i}^{f_2}\sigma \}_{i=1}^{l}, \Omega = e(g, h)^\gamma \right).
\]

**Encrypt(\(x, M, PK\))**: The encrypt algorithm takes as input a vector \( x = (x_1, \ldots, x_l) \in \Sigma_*^l \), a message \( M \in \mathcal{M} \subseteq \mathbb{G}_T \) and the public key \( PK \). It first chooses a random exponent \( t, \alpha, \beta \in \mathbb{Z}_m \) and random elements \( \{R_{3,i}, R_{4,i}\}_{i=1}^{l} \in \mathbb{G}_r \). Next, it outputs a ciphertext as

\[
CT = \left( C = \Omega^tM, C_0 = g^{\gamma}, \{ C_{1,i} = H_{1,i}^{\alpha}, Q^{\alpha \sigma}R_{3,i}, C_{2,i} = H_{2,i}^{\beta}, Q^{\beta \sigma}R_{4,i} \}_{i=1}^{l} \right).
\]

**Query(\(CT, TK_{\sigma}, PK\))**: The query algorithm takes as input a ciphertext \( CT \) and a token \( TK_{\sigma} \) of a vector \( \sigma \). It first computes

\[
M \leftarrow C \cdot e(C_0, K_0) \cdot \prod_{i=1}^{l}(e(C_{1,i}, K_{1,i}) \cdot e(C_{2,i}, K_{2,i})).
\]

If \( M \notin \mathcal{M} \), it outputs \( \perp \) indicating that the predicate \( f_{\sigma} \) is not satisfied. Otherwise, it outputs \( M \) indicating that the predicate \( f_{\sigma} \) is satisfied.

**Theorem 3.3.1 (\[24\])**. The IPE scheme of Katz, Sahai, and Waters is selectively secure under the two static assumptions.
3.3.2 Conversion from IPE to HVE

The hidden vector encryption $\mathcal{HVE} = \langle \text{Setup, KeyGen, Encrypt, Query} \rangle$ using inner product encryption $\mathcal{IPE} = \langle \text{Setup}', \text{KeyGen}', \text{Encrypt}', \text{Query}' \rangle$ is described as follows.

**Setup($1^\lambda, l$):** The setup algorithm first run $\text{Setup}'(1^\lambda, 2l)$ and obtains $(\text{PK}', \text{SK}')$. Then it keeps $\text{SK}'$ as a secret key and publishes a public key as $\text{PK} = \text{PK}'$.

**GenToken($\sigma, \text{SK}, \text{PK}$):** The token generation algorithm takes as input a vector $\sigma = (\sigma_1, \ldots, \sigma_l) \in \Sigma^l$, the secret key $\text{SK}$, and the public key $\text{PK}$. It first convert the vector $\sigma$ to a vector $\sigma' = (\sigma'_1, \ldots, \sigma'_{2l}) \in \Sigma^{2l}$ as follows:

- if $\sigma_i \neq \ast$, then $\sigma'_{2i-1} = 1$, $\sigma'_{2i} = \sigma_i$.
- if $\sigma_i = \ast$, then $\sigma'_{2i-1} = 0$, $\sigma'_{2i} = 0$.

Then it outputs a token obtained by running $\text{GenToken}'(\sigma', \text{SK}', \text{PK}')$.

**Encrypt($x, M, \text{PK}$):** The encrypt algorithm takes as input a vector $x = (x_1, \ldots, x_l) \in \Sigma^l$, a message $M$ and the public key $\text{PK}$. It first choose random values $r_1, \ldots, r_l \in \Sigma$ and construct a vector $x' = (x'_1, \ldots, x'_{2l})$ as follows:

$$x'_{2i-1} = -r_i \cdot x_i, \quad x'_{2i} = r_i.$$ 

It outputs a ciphertext by running $\text{Encrypt}'(x', M, \text{PK}')$.

**Query($CT, TK_\sigma, \text{PK}$):** The query algorithm outputs $\text{Query}'(CT, TK_\sigma, \text{PK})$. 
Chapter 4

Efficient HVE with Short Tokens

4.1 Overview

In this chapter, we construct efficient HVE schemes that have short tokens and prove their selective model security under simple assumptions. Our constructions are algebraically similar to the one of Shi and Waters, but the cost of decrypt operation in our constructions is constant.

Previous research on HVE has mainly focused on improving the expressiveness of predicates or providing additional properties like the delegation. To apply HVE schemes to real applications, it is important to construct an efficient HVE scheme. One can measure the efficiency of HVE in terms of the ciphertext size, the token size, and the number of pairing operations in decryption. Let \( l \) be the number of attributes in the ciphertext and \( s \) be the number of attributes in the token except the wild card attribute. Then the efficiency of previous HVE schemes is compared in Table 4.1. Theoretically, the number of group elements in ciphertext should be proportional to the number of attributes in the ciphertexts, so the minimum size of ciphertext is \( l|G| + O(1) \). However, the token size and the number of pairing operations in decryption can be constant, that is, independent of \( l \). Therefore constructing an HVE scheme with the constant size of tokens and the constant number of pairing operations is an important problem to solve.

We propose HVE schemes that have the constant size of tokens and the constant cost of pairing operations. Our first construction is based on composite order bilinear groups whose order is a product of three primes. The ciphertext consists of \( l + O(1) \) group elements, the token consists of four group elements, and the decryption requires four pairing computations. Our second one is based on prime order asymmetric bilinear groups where isomorphisms between two groups are not efficiently computable.

Though our construction in composite order bilinear groups is algebraically similar to the one by Shi and Waters in [36], we achieved the constant size of tokens and the constant cost of decryption, in contrast to the construction of Shi and Waters. The main technique for our constructions is to use the same random value for each attributes in the token. In contrast, the construction of Shi and Waters used different random values for each attributes. This technique is reminiscent of the one that enables the design of HIBE with the constant size of ciphertexts in [6]. However, it is not easy to prove the security of HVE when the same random value is used in the token, since HVE should provide an additional security property, namely attribute hiding, that is, the ciphertext does not reveal any information about the attributes.
Table 4.1: Comparison between previous HVE schemes and ours

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Group Order</th>
<th>Ciphertext Size</th>
<th>Token Size</th>
<th>No. of Pairing</th>
</tr>
</thead>
<tbody>
<tr>
<td>BW-HVE [13]</td>
<td>pq</td>
<td>$2l</td>
<td>G</td>
<td>+ O(1)$</td>
</tr>
<tr>
<td>SW-dHVE [36]</td>
<td>pqr</td>
<td>$l</td>
<td>G</td>
<td>+ O(1)$</td>
</tr>
<tr>
<td>IP-HVE [23]</td>
<td>p</td>
<td>$2l</td>
<td>G</td>
<td>+ O(1)$</td>
</tr>
<tr>
<td>OT-IPE [29]</td>
<td>p</td>
<td>$2l</td>
<td>G</td>
<td>+ O(1)$</td>
</tr>
<tr>
<td>Ours</td>
<td>pqr</td>
<td>$l</td>
<td>G</td>
<td>+ O(1)$</td>
</tr>
</tbody>
</table>

$p, q, r$ are prime values, $l$ = no. of attributes in ciphertext, $s$ = no. of attributes in token.

4.2 HVE in Composite Order Groups

In this section, we construct an HVE scheme based on composite order bilinear groups and prove security under the decisional cBDH, BSD, and decisional C3DH assumptions. Our construction has a similar algebraic structure to the construction of Shi and Waters [36], but ours has the constant size of tokens and the constant number of pairing operations.

4.2.1 Construction

Let $\Sigma = \mathbb{Z}_m$ for some integer $m$ and set $\Sigma_s = \mathbb{Z}_m \cup \{\ast\}$. Our scheme is described as follows.

Setup$(\lambda)$: The setup algorithm first generates the bilinear group $G$ of composite order $n = pqr$ where $p, q$ and $r$ are random primes of bit size $\Theta(\lambda)$ and $p, q, r > m$. Next, it chooses random elements $v, w_1, w_2 \in \mathbb{G}_p$, $(u_1, h_1), \ldots, (u_l, h_l) \in \mathbb{G}_p^2$, and exponents $\alpha, \beta \in \mathbb{Z}_p$. It keeps these as a secret key $SK$. Then it chooses random elements $R_v, R_{w_1,1}, R_{w_2,1} \in \mathbb{G}_q$ and $(R_{u_1,1}, R_{h_1,1}), \ldots, (R_{u_l,1}, R_{h_l,1}) \in \mathbb{G}_p^2$, and it publishes a public key $PK$ with the description of the bilinear group $G$ as follows

$$PK = \left( V = vR_v, \ W_1 = w_1R_{w_1,1}, \ W_2 = w_2R_{w_2,1}, \ \{ (U_i = u_iR_{u,i}, \ H_i = h_iR_{h,i}) \}_{i=1}^{l} \right),$$

$$g_q, g_r, \ \Omega = e(v, g)^{\alpha \beta}.$$

Gentoken($\sigma, SK, PK$): The token generation algorithm takes as input a vector $\sigma = (\sigma_1, \ldots, \sigma_l) \in \Sigma_s^l$ and the secret key $SK$. It first selects random exponents $r_1, r_2, r_3 \in \mathbb{Z}_p$ and random elements $Y_0, Y_1, Y_2, Y_3 \in \mathbb{G}_p$ by raising $g_r$ to random exponents in $\mathbb{Z}_n$. Let $S$ be the set of indexes that are not wild card positions in the vector $\sigma$. Then it outputs a token as

$$TK_\sigma = \left( K_0 = g^{\alpha \beta}v^{r_1}w_1^{r_2} \prod_{i \in S} u_i^{\sigma_i}h_i^{r_3}Y_0, \ K_1 = v^{r_1}Y_1, \ K_2 = v^{r_2}Y_2, \ K_3 = v^{r_3}Y_3 \right).$$

Encrypt($x, M, PK$): The encrypt algorithm takes as input a vector $x = (x_1, \ldots, x_l) \in \Sigma_s^l$, a message $M \in \mathcal{M} \subseteq \mathbb{G}_T$ and the public key $PK$. It first chooses a random exponent $t \in \mathbb{Z}_n$ and random elements $Z_0, Z_1, Z_2, Z_{3,1}, \ldots, Z_{3,l} \in \mathbb{G}_q$ by raising $g_q$ to random elements from $\mathbb{Z}_n$. Next, it outputs a ciphertext as

$$CT = \left( C = \Omega^tM, \ C_0 = V^tZ_0, \ C_1 = W_1^tZ_1, \ C_2 = W_2^tZ_2, \ \{ C_{3,i} = (U_i^tH_i)^tZ_{3,i} \}_{i=1}^{l} \right).$$
Query(CT, TK_\sigma, PK): The query algorithm takes as input a ciphertext CT and a token TK_\sigma of a vector \sigma. It first computes

\[ M \leftarrow C \cdot e(C_0, K_0)^{-1} \cdot e(C_1, K_1) \cdot e(C_2, K_2) \cdot e(\prod_{i \in S} C_{3,i}, K_3). \]

If \( M \notin \mathcal{M} \), it outputs \( \perp \) indicating that the predicate \( f_\sigma \) is not satisfied. Otherwise, it outputs \( M \) indicating that the predicate \( f_\sigma \) is satisfied.

**Remark 3.** In our construction, we limited the finite set \( \Sigma \) of attributes to be \( \mathbb{Z}_m \). If we use a collision-resistant hash function, then we can easily expand this space to all of \( \{0,1\}^* \) when \( m \) is large enough to contain the range of the hash function.

### 4.2.2 Correctness

If \( f_\sigma(x) = 1 \), then the following simple calculation shows that Query(CT, TK_\sigma, PK) = M as

\[
e(C_0, K_0)^{-1} \cdot e(C_1, K_1) \cdot e(C_2, K_2) \cdot e(\prod_{i \in S} C_{3,i}, K_3)
= e(v', g^{\alpha \beta} w_1^1 w_2^2 \prod_{i \in S} u_i^3 h_i)^{r_3} \cdot e(w_1^1, v'^1) \cdot e(w_2^2, v'^2) \cdot e(\prod_{i \in S} (u_i^3 h_i)^l, v'^{l})
= e(v', g^{\alpha \beta})^{-1} \cdot e(\prod_{i \in S} (u_i^3)^{-\sigma_i + x_i} h_i, v') = e(v', g^{\alpha \beta})^{-1}.
\]

Otherwise, that is \( f_\sigma(x) = 0 \), then we can use Lemma 5.2 in [13] to show that the probability of Query(CT, TK_\sigma, PK) \( \neq \perp \) is negligible by limiting |\( \mathcal{M} \)| to less than |\( \mathbb{G}_T \)|\(^{1/4}\).

### 4.2.3 Security

**Theorem 4.2.1.** The above HVE construction is selectively secure under the decisional cBDH assumption, the BSD assumption, and the decisional C3DH assumption.

**Proof.** Suppose there exists an adversary that distinguishes the original selective security game. Then the adversary commits two vectors \( x_0 = (x_{0,1}, \ldots, x_{0,l}) \) and \( x_1 = (x_{1,1}, \ldots, x_{1,l}) \in \Sigma^l \) at the beginning of the game. Let \( X \) be the set of indexes \( i \) such that \( x_{0,i} = x_{1,i} \) and \( X \) be the set of indexes \( i \) such that \( x_{0,i} \neq x_{1,i} \). The proof uses a sequence of four games to argue that the adversary cannot win the original security game. Each individual game is described as follows.

**Game_0.** This game denotes the original selective security game that is defined in Section 2.1.

**Game_1.** We first modify Game_0 slightly into a new game Game_1. Game_1 is almost identical to Game_0 except in the way the challenge ciphertext elements are generated. In Game_1, if \( M_0 \neq M_1 \), then the simulator generates the challenge ciphertext element \( C \) by multiplying a random element in \( \mathbb{G}_T \) and it generates the rest of the ciphertext elements as usual. If \( M_0 = M_1 \), then the challenge ciphertext is generated correctly.

**Game_2.** Next, we modify Game_1 into a new game Game_2. Game_2 is almost identical to Game_1 except in the way the tokens are generated. Let \( S \) be the set of indexes that are not wild card positions of the token query vector \( \sigma \). Then any token query by the adversary must satisfy one of the following two cases:

- Type 1 \( f_\sigma(x_0) = f_\sigma(x_1) = 1 \). In this case, \( S \cap X = \emptyset \) and \( \sigma_j = x_{0,j} = x_{1,j} \) for all index \( j \in S \cap X \).
• Type 2 \( f_\sigma(x_0) = f_\sigma(x_1) = 0 \). In this case, there exists an index \( j \in S \) such that \( \sigma_j \neq x_{\gamma,j} \) for all \( \gamma \in \{0, 1\} \).

In Game2, if the adversary requests the Type 1 token query, then the simulator chooses two exponents \( r_1 \) and \( r_2 \) not independently at random, but in a correlated way as \( r_1 = \pi r_2 \) for a fixed value \( \pi \). The simulator can use this correlation to simulate this game. However, the adversary cannot distinguish this correlation because of random blinding elements \( \mathbb{G}_r \) in the token.

**Game3.** We modify Game2 into a game Game3. Game2 and Game3 are identical except in the challenge ciphertext. In Game3, the simulator creates the ciphertext according to the following distribution as

\[
C_1 = W_1^i g_p^{\rho} Z_1, \quad C_2 = W_2^j g_p^{\rho \pi} Z_2,
\]

where \( \rho \) is a random value in \( \mathbb{Z}_p \) and \( \pi \) is the fixed value in \( \mathbb{Z}_p \) but \( \pi \) is hidden from the adversary.

**Game4.** We now define a new game Game4. Game4 differs from Game3 in that for all \( i \in \mathcal{X} \), the ciphertext component \( C_i \) is replaced by a random element from \( \mathbb{G}_{pq} \). Note that in Game4, the ciphertext gives no information about the vector \( x_f \) or the message \( M_T \) encrypted. Therefore, the adversary can win Game4 with probability at most \( 1/2 \).

Through the following four lemmas, we will prove that it is hard to distinguish Game\(_{i-1}\) from Game\(_i\) under the given assumptions. Thus, the proof is easily obtained by the following four lemmas. This completes our proof.

**Lemma 4.2.2.** If the decisional cBDH assumption and the BSD assumption hold, then no polynomial-time adversary can distinguish between Game0 and Game1 with a non-negligible advantage.

**Proof.** For this lemma, we additionally define a sequence of games Game\(_{0,0}\), Game\(_{0,1}\), and Game\(_{0,2}\) where Game\(_{0,0}\) = Game0. Game\(_{0,1}\) and Game\(_{0,2}\) are almost identical to Game\(_{0,0}\) except in the way the challenge ciphertext is generated. In Game\(_{0,1}\), if \( M_0 \neq M_1 \), then the simulator generates the challenge ciphertext element \( C_i \) by multiplying a random element in \( \mathbb{G}_{T,p} \), and it generates the rest of the ciphertext elements as usual. If \( M_0 = M_1 \), then the challenge ciphertext is generated correctly. In Game\(_{0,2}\), if \( M_0 \neq M_1 \), then the simulator generates the challenge ciphertext element \( C_i \) as a random element from \( \mathbb{G}_T \) instead of \( \mathbb{G}_{T,p} \), and it generates the rest of the ciphertext elements as usual. If \( M_0 = M_1 \), then the challenge ciphertext is generated correctly. It is not hard to see that Game\(_{0,2}\) is identical to Game1.

Suppose there exists an adversary \( \mathcal{A} \) that distinguishes between Game\(_{0,0}\) and Game\(_{0,1}\) with a non-negligible advantage. A simulator \( \mathcal{B} \) that solves the decisional cBDH assumption using \( \mathcal{A} \) is given: a challenge tuple \( D = ((p, q, r, \mathbb{G}, \mathbb{G}_T, e), g_p, g_q, g_r, g_p^a, g_p^b, g_p^c) \) and \( T \) where \( T = e(g_p, g_p)^{abc} \) or \( T = R \in \mathbb{G}_{T,p} \). Then \( \mathcal{B} \) that interacts with \( \mathcal{A} \) is described as follows.

**Init:** \( \mathcal{A} \) gives two vectors \( x_0 = (x_0,1, \ldots, x_0,\ell), x_1 = (x_1,1, \ldots, x_1,\ell) \in \mathcal{X}^\ell \). \( \mathcal{B} \) then flips a random coin \( \gamma \) internally.

**Setup:** \( \mathcal{B} \) first chooses random elements \( R_u, R_{u,1}, R_{u,2} \in \mathbb{G}_q \), \( (R_{u,1}, R_{h,1}), \ldots, (R_{u,t}, R_{h,1}) \in \mathbb{G}_q^2 \), and random exponents \( v', w'_1, w'_2 \in \mathbb{Z}_q \), \( (u'_1, h'_1^1), \ldots, (u'_1, h'_1^\ell) \in \mathbb{Z}_q^\ell \). Next, it publishes the group description \( (n, \mathbb{G}, \mathbb{G}_T, e) \) and a public key as

\[
V = g_p^{v'}, W_1 = g_p^{w'_1} R_{u,1}, W_2 = g_p^{w'_2} R_{u,2}, \quad \{(U_i = (g_p^a)^{u'_i} R_{u,i}); \quad H_i = g_p^{h'_i} (g_p^a)^{-u'_i \gamma} / R_{h,i}\},
\]

\( g_q, g_r, \Omega = e(g_p^a, g_p^b)^{v'} \).

24
Query 1: A adaptively requests a token for a vector $\sigma = (\sigma_1, \ldots, \sigma_l) \in \Sigma^l$ to $B$. Let $S$ be the set of indexes that are not wild card positions.

Type 1 If $A$ requests a Type 1 query, then $B$ simply aborts and takes a random guess. The reason for this is by our definition such as if a Type 1 query is made then the challenge messages $M_0, M_1$ will be equal. However, in this case the games $\text{Game}_0$ and $\text{Game}_1$ are identical, so there can be no difference in the adversary’s advantage.

Type 2 If $A$ requests a Type 2 query, then there exists an index $j \in S$ such that $\sigma_j \neq x_{\gamma,j}$. Let $\Delta = \sum_{i \in S} u_i^l(\sigma_i - x_{\gamma,i}) \in \mathbb{Z}_p$. Note that $\Delta \neq 0$ except with negligible probability. If $\Delta \neq 0$, then $B$ chooses random exponents $r'_1, r'_2, r'_3 \in \mathbb{Z}_p$ and random elements $Y_0, Y_1, Y_2, Y_3 \in \mathbb{G}_r$. Next, it creates a token as

$$K_0 = g_{p,r'_1}^{w_i', r_1^l} g_{p,r'_2}^{w_i', r_2^l} (g_p^h)^{-\sum_{i \in S} h_i^l/\Delta} \prod_{i \in S} ((g_p^a)^{u_i^l(\sigma_i - x_{\gamma,i})} g_p^b)^{h_i^l} r_i^l Y_0,$$

$$K_1 = g_{p,r'_1}^{h_i^l} Y_1, \quad K_2 = g_{p,r'_2}^{h_i^l} Y_2, \quad K_3 = g_{p,r'_3}^{h_i^l} (g_p^h)^{-\Delta/3} Y_3.$$

Note that it can compute $\Delta^{-1}$ since it knows $p$. To show that the above token is the same as the real scheme, we define the randomness of the token as

$$r_1 = r'_1 \mod p, \quad r_2 = r'_2 \mod p, \quad r_3 = r'_3 - b/\Delta \mod p.$$

It is obvious that $r_1, r_2, r_3$ are all uniformly distributed if $r'_1, r'_2, r'_3$ are independently chosen at random. The following calculation shows that the above token is correctly distributed as the token in the real scheme as

$$K_0 = g_{p,r'_1}^{w_i', r_1^l} g_{p,r'_2}^{w_i', r_2^l} (g_p^h)^{-\sum_{i \in S} h_i^l/\Delta} \prod_{i \in S} ((g_p^a)^{u_i^l(\sigma_i - x_{\gamma,i})} g_p^b)^{h_i^l} r_i^l Y_0$$

$$= g_{p,r'_1}^{w_i', r_1^l} g_{p,r'_2}^{w_i', r_2^l} (g_p^h)^{-\sum_{i \in S} h_i^l/\Delta} \prod_{i \in S} ((g_p^a)^{u_i^l(\sigma_i - x_{\gamma,i})} g_p^b)^{h_i^l} r_i^l Y_0.$$

Challenge: $A$ gives two messages $M_0, M_1$ to $B$. If $M_0 = M_1$, then $B$ aborts and takes a random guess. Otherwise, it chooses random elements $Z_0, Z_1, Z_2, Z_3, \ldots, Z_{3,l} \in \mathbb{G}_q$ and outputs a challenge ciphertext as

$$C = T^{\gamma'} M_T, \quad C_0 = (g_c)^{\gamma'} Z_0, \quad C_1 = (g_c)^{\gamma'} Z_1, \quad C_2 = (g_c)^{\gamma'} Z_2, \quad \forall i : C_{3,i} = (g_c)^{\gamma'} Z_{3,i}.$$

If $T$ is a valid cBDH tuple, then $B$ is playing $\text{Game}_{0,0}$. Otherwise, it is playing $\text{Game}_{0,1}$.

Query 2: Same as Query Phase 1.

Guess: $A$ outputs a guess $\gamma'$. If $\gamma = \gamma'$, it outputs 0. Otherwise, it outputs 1.

Suppose there exists an adversary $A$ that distinguishes between $\text{Game}_{0,1}$ and $\text{Game}_{0,2}$ with a non-negligible advantage. A simulator $B$ that solves the BSD assumption using $A$ is given: a tuple $D = ((n, \mathbb{G}, \mathbb{G}_T, e), g_p, g_q, g_r)$ and $T$ where $T = Q \in \mathbb{G}_{T,p}$ or $T = R \in \mathbb{G}_T$. Then $B$ that interacts with $A$ is described as follows.

Init: $A$ gives two vectors $x_0, x_1 \in \Sigma^l$. $B$ then flips a random coin $\gamma$ internally.
Setup: $B$ sets up the public key as the real setup algorithm using $g_p, g_q, g_r$ from the assumption.

Query 1: $B$ answers token queries by running the real token generation algorithm except that it chooses random exponents from $\mathbb{Z}_n$ instead of $\mathbb{Z}_p$. However, this does not affect the simulation since it will raise the elements from $\mathbb{G}_p$ to the exponents.

Challenge: $A$ gives two messages $M_0, M_1$ to $B$. If $M_0 = M_1$, then $B$ encrypts the message to the vector $x_r$. Otherwise, it creates the challenge ciphertext of message $M_p$ to $x_r$ as normal with except that $C$ is multiplied by $T$. If $T \in \mathbb{G}_{T,p}$, then $B$ is playing $\text{Game}_{0,1}$. Otherwise, it is playing $\text{Game}_{0,2}$.

Query 2: Same as Query Phase 1.

Guess: $A$ outputs a guess $\gamma'$. If $\gamma = \gamma'$, it outputs 0. Otherwise, it outputs 1.

This completes our proof. \(\square\)

Lemma 4.2.3. If the decisional $C3DH$ assumption holds, then no polynomial-time adversary can distinguish between $\text{Game}_1$ and $\text{Game}_2$ with a non-negligible advantage.

Proof. Let $q_1$ denote the maximum number of Type 1 queries made by the adversary. We define a sequence of games $\text{Game}_{1,0}, \text{Game}_{1,1}, \ldots, \text{Game}_{1,q_1}$ where $\text{Game}_{1,0} = \text{Game}_1$. In $\text{Game}_{1,i}$, for all $k$-th Type-1 queries such that $k > i$, the simulator creates the token as usual using three independent random exponents $r_1, r_2, r_3 \in \mathbb{Z}_n$. However, for all $k$-th Type-1 queries such that $k \leq i$, the simulator creates token components using the correlated random exponents such that $r_1 = \pi r_2$ for a fixed value $\pi$. It is obvious that $\text{Game}_{1,q_1}$ is equal with $\text{Game}_2$.

Before proving this lemma, we introduce the decisional Composite 2-party Diffie-Hellman (C2DH) assumption as follows: Let $(n, G, G_T, e)$ be a description of the bilinear group of composite order $n = pqr$. Let $g_p, g_q, g_r$ be generators of subgroups of order $p, q, r$ of $G$ respectively. The decisional C2DH problem is stated as follows: given a challenge tuple $D = ((n, G, G_T, e), g_p, g_q, g_r, g_p^a R_1, g_p^b R_2)$ and $T$, decides whether $T = g_p^{ab} R_3$ or $T = R$ with random choices of $R_1, R_2, R_3 \in \mathbb{G}_q, R \in \mathbb{G}_{pq}$. It is easy to show that if there exists an adversary that breaks the decisional C2DH assumption, then it can break the decisional C3DH assumption.

Suppose there exists an adversary $A$ that distinguishes between $\text{Game}_{1,d-1}$ and $\text{Game}_{1,d}$ with a non-negligible advantage. A simulator $B$ that solves the decisional C2DH assumption using $A$ is given: a challenge tuple $D = ((n, G, G_T, e), g_p, g_q, g_r, g_p^a Y_1, g_p^b Y_2)$ and $T$ where $T = g_p^{ab} Y_3$ or $T = R$ with random choices of $Y_1, Y_2, Y_3 \in \mathbb{G}_r, R \in \mathbb{G}_{pr}$. Then $B$ that interacts with $A$ is described as follows.

Init: $A$ gives two vectors $x_0, x_1 \in \Sigma^l$. $B$ then flips a random coin $\gamma$ internally.

Setup: $B$ first chooses random exponents $v', w_1', w_2', \alpha, \beta \in \mathbb{Z}_n$, $(u_1', h_1'), \ldots, (u_l', h_l') \in \mathbb{Z}_n^2$, then it sets $v = g_p^{v'}, w_1 = g_p^{w_1'}, w_2 = g_p^{w_2'}, u_i = g_p^{u_i'}, h_i = g_p^{h_i'}$. Next, it chooses random elements $R_v, R_{w_1}, R_{w_2} \in \mathbb{G}_q, (R_{u_1}, R_{h_1}), \ldots, (R_{u_l}, R_{h_l}) \in \mathbb{G}_q^2$, and it publishes the group description and a public key as

$$V = v R_v, W_1 = w_1 R_{w_1}, W_2 = w_2 R_{w_2}, \{(U_i = u_i R_{u,i}, H_i = h_i R_{h,i})\}, g_q, g_r, \Omega = e(v, g_p)\alpha\beta.$$

Query 1: $A$ adaptively requests a token for a vector $\sigma = (\sigma_1, \ldots, \sigma_l) \in \Sigma^l$ to $B$. Let $S$ be the set of indexes that are not wild card positions.
Type 1 Let $k$ be the index of Type 1 queries. If $\mathcal{A}$ requests a Type 1 query, then $\mathcal{B}$ chooses random exponents $r_1, r_2, r_3 \in \mathbb{Z}_n$ and random elements $Y_0, Y_1, Y_1', Y_2, Y_3' \in \mathbb{G}_t$. Next, it creates a token depending on the $k$ value as

$$
k < d : K_0 = g_p^{\alpha \beta} ((g_p^a Y_1)^{w_1} Y_0) (\prod_{i \in S} u_i^{a_i h_i})^{r_3} Y_0', K_1 = (g_p^a Y_1)^{v_1} Y_1', K_2 = v_2 Y_1', K_3 = v_3 Y_3',$$

$$
k = d : K_0 = g_p^{\alpha \beta} T^{w_1} ((g_p^b Y_2)^{w_2} (\prod_{i \in S} u_i^{a_i h_i})^{r_3} Y_0', K_1 = T^{v_1} Y_1', K_2 = (g_p^b Y_2)^{v_2} Y_1', K_3 = v_3 Y_3',$$

$$
k > d : K_0 = g_p^{\alpha \beta} w_1^{r_1} w_2^{r_2} (\prod_{i \in S} u_i^{a_i h_i})^{r_3} Y_0, K_1 = (g_p^{ab} Y_2)^{v_1} Y_1 = v^{ab} Y_1 = v^{\pi r_2} Y_1, K_2 = (g_p^{ab} Y_2)^{v_2} Y_2 = v^{b} Y_2 = v^{b} Y_1',$$

where $\pi = a$ and $r_2 = b$.

Type 2 If $\mathcal{A}$ requests a Type 2 query, then $\mathcal{B}$ creates the token as the real token generation algorithm since it knows all values that are needed.

Challenge: $\mathcal{A}$ gives two messages $M_0, M_1$ to $\mathcal{B}$. $\mathcal{B}$ creates the ciphertext for $M_0$ and $x_0$ as the real encrypt algorithm by choosing a random exponent $t \in \mathbb{Z}_n$ and random elements in $\mathbb{G}_q$.

Query 2: Same as Query Phase 1.

Guess: $\mathcal{A}$ outputs a guess $\gamma'$. If $\gamma = \gamma'$, it outputs 0. Otherwise, it outputs 1.

This completes our proof.

Lemma 4.2.4. If the decisional C3DH assumption holds, then no polynomial-time adversary can distinguish between Game2 and Game3 with a non-negligible advantage.

Proof: Suppose there exists an adversary $\mathcal{A}$ that distinguishes between Game2 and Game3 with a non-negligible advantage. A simulator $\mathcal{B}$ that solves the decisional C3DH assumption using $\mathcal{A}$ is given: a challenge tuple $D = ((n, \mathbb{G}, G_T, e), g_p, g_q, g_r, g_p^a, g_p^b, g_p^{ab} R_1, g_p^{abc} R_2)$ and $T$ where $T = g_p^a R_3$ or $T = g_p^b R_3$ for a random exponent $d \in \mathbb{Z}_n$. Then $\mathcal{B}$ that interacts with $\mathcal{A}$ is described as follows.

Init: $\mathcal{A}$ gives two vectors $x_0 = (x_{0,1}, \ldots, x_{0,l})$, $x_1 = (x_{1,1}, \ldots, x_{1,l}) \in \Sigma^l$. $\mathcal{B}$ then flips a random coin $\gamma$ internally.

Setup: $\mathcal{B}$ first chooses random exponents $w_1, w_2, \alpha, \beta \in \mathbb{Z}_n$, $(u_1', h_1'), \ldots, (u_l', h_l') \in \mathbb{Z}_n^2$, and random elements $R_v, R_{w_1}, R_{w_2} \in \mathbb{G}_q$, $(R_{u_1}, R_{h_1}), \ldots, (R_{u_l}, R_{h_l}) \in \mathbb{G}_q^2$. Next, it publishes a public key as

$$V = (g_p^{ab} R_1) R_v, W_1 = (g_p^{ab} R_1 \cdot g_p) w_1 R_{w_1}, W_2 = g_p w_2 R_{w_2}, \{ (U_i = (g_p^{ab} u_i R_{u_i}, H_i = (g_p^a)^{-u_i x_{i}} (g_p^{ab} R_1)^{h_i R_{h_i}} ) \} \leq i \leq l, g_q, \Omega = e(g_p^a R_1, g_p^b)^{\alpha \beta}.$$

27
Query 1: A adaptively requests a token for a vector $\sigma = (\sigma_1, \ldots, \sigma_l) \in \Sigma^l$ to $B$. Let $S$ be the set of indexes that are not wild card positions.

**Type 1** If $A$ requests a Type 1 query, then $B$ chooses random exponents $r'_1, r'_3 \in \mathbb{Z}_n$ and random elements $Y_0, Y_1, Y_2, Y_3 \in \mathbb{G}_r$. Next, it creates a token as

$$K_0 = g_p^{\alpha \beta} (g_p^{a})^{w'_1 w'_3} g_p^{\sum_{i \in S} b'_i} Y_0, K_1 = (g_p^{a})^{w'_1} Y_1, K_2 = (g_p^{a})^{-w'_1 r'_3} Y_2, K_3 = g_p^{r'_1} Y_3.$$  

To show that the above token is the same as the token in Game$_3$, we define the randomness of the token as

$$r_1 = w'_2 r'_1 / b \mod p, \quad r_2 = -w'_1 r'_3 / b \mod p, \quad r_3 = r'_3 / ab \mod p.$$  

It is obvious that two random $r_1$ and $r_2$ are correlated as $r_1 = \pi r_2$ where $\pi = -w'_2 / w'_1$. The distribution of the above token is correct as follows

$$K_0 = g_p^{\alpha \beta} (g_p^{(ab+1)w'_1})^{w'_1 r'_1 / b} (g_p^{a})^{w'_1} g_p^{\sum_{i \in S} (b'_i \alpha - x_i r'_1 + (ab) w'_1 r'_3 / b)} Y_0$$

$$= g_p^{\alpha \beta} g_p^{aw'_1 r'_1} g_p^{\sum_{i \in S} b'_i r'_1} Y_0.$$  

**Type 2** If $A$ requests a Type 2 query, then there exists an index $j \in S$ such that $\sigma_j \neq x_{r,j}$. Let $\Delta = \sum_{i \in S} (\sigma_i - x_{r,i}) \in \mathbb{Z}_p$. Note that $\Delta \neq 0$ except with negligible probability. $B$ first chooses random exponents $r'_1, r'_2, r'_3 \in \mathbb{Z}_n$ and random elements $Y_0, Y_1, Y_2, Y_3 \in \mathbb{G}_r$, then it creates a token as

$$K_0 = g_p^{\alpha \beta} (g_p^{a})^{w'_1 w'_3} g_p^{\Delta w'_1 r'_3} g_p^{\sum_{i \in S} h_i' w'_1 r'_i} g_p^{\sum_{i \in S} h_i' w'_1 r'_i} Y_0,$$

$$K_1 = (g_p^{a})^{w'_1 r'_1} Y_1, K_2 = (g_p^{a})^{-w'_1 r'_1} Y_2, K_3 = g_p^{a} Y_3.$$  

To show that the above token is the same as the token in Game$_3$, we define the randomness of the token as

$$r_1 = w'_2 r'_1 / b \mod p, \quad r_2 = -w'_1 r'_3 / b - b r'_2 / ab \mod p,$$

$$r_3 = w'_2 r'_2 / b + w'_3 r'_2 / ab \mod p.$$  

It is not hard to see that $r_1, r_2, r_3$ are independent random values since $\Delta \neq 0$ except with negligible probability. The distribution of the above token is correct as follows

$$K_0 = g_p^{\alpha \beta} (g_p^{(ab+1)w'_1})^{w'_1 r'_1 / b} g_p^{w'_1 r'_1 / b - \sum_{i \in S} (\sigma_i - x_{r,i}) r'_2 / ab} g_p^{\sum_{i \in S} (b'_i \alpha - x_{r,i} + (ab) w'_1 r'_3 / b + w'_2 r'_2 / ab)} Y_0$$

$$= g_p^{\alpha \beta} g_p^{aw'_1 r'_1} g_p^{\Delta w'_1 r'_3} g_p^{\sum_{i \in S} (\sigma_i' w'_1 r'_i + h_i' w'_1 r'_i)} Y_0.$$  

**Challenge:** $A$ gives two messages $M_0, M_1$ to $B$. If $M_0 = M_1$, then $B$ computes $C = e(g_p^{abc} R_2, g_p^{\alpha \beta} M_7)$. Otherwise, it chooses a random elements in $\mathbb{G}_T$ for $C$. Next, it chooses random elements $Z_0, Z_1, Z_2, Z_3, \ldots$, $Z_{3, j} \in \mathbb{G}_q$ and outputs a challenge ciphertext as

$$C_0 = (g_p^{abc} R_2) Z_0, C_1 = (g_p^{abc} R_2 \cdot T)^{w'_1} Z_1, C_2 = T^{w'_1} Z_2, \forall i : C_{3,i} = (g_p^{abc} R_2)^{w'_i} Z_{3,i}.$$  

If $T$ is a valid C3DH tuple, then $B$ is playing Game$_2$. Otherwise, it is playing Game$_3$ as follows

$$C_1 = (g_p^{abc} R_2 \cdot g_p^{d} R_3)^{w'_1} Z_1 = (g_p^{abc} g_p^{c-e} g_p^{d})^{w'_1} Z_1 = (g_p^{abc} g_p^{e - c + d})^{w'_1} Z_1 = W_1 g_p^{d} Z'_1,$$

$$C_2 = (g_p^{d} R_3)^{w'_1} Z_2 = g_p^{\rho (p/w'_1 + c)}^{w'_1} Z_2 = g_p^{c} g_p^{p w'_1}^{w'_1} Z_2 = W_2 g_p^{p - \rho \pi} Z'_2$$

where $T = g_p^{d} R_3$, $\rho = (-c + d) w'_1$ and $\pi = -w'_2 / w'_1$.  

28
Query 2: Same as Query Phase 1.

Guess: \( A \) outputs a guess \( \gamma' \). If \( \gamma = \gamma' \), it outputs 0. Otherwise, it outputs 1.

This completes our proof. \( \square \)

Lemma 4.2.5. If the decisional C3DH assumption holds, then no polynomial-time adversary can distinguish between Game_3 and Game_4 with a non-negligible advantage.

Proof. Let \( X \) denote the set of indexes \( i \) where two committed vectors \( x_0, x_1 \) are not equal. We define a sequence of games \( \text{Game}_{3,0}, \text{Game}_{3,1}, \ldots, \text{Game}_{3,|X|} \) where \( \text{Game}_{3,0} = \text{Game}_3 \). Let \( X_i \subseteq X \) denote the set of first \( i \) indexes in \( X \). In \( \text{Game}_{3,i} \), the simulator creates ciphertext elements \( C_i, C_0, \) and \( C_j \) normally for all \( j \notin X_i \). For all \( j \in X_i \), the simulator replaces \( C_j \) with random elements in \( \mathbb{G}_{pq} \). For \( C_1, C_2 \), the simulator creates the following ciphertext elements like in game \( \text{Game}_4 \) as

\[
C_1 = W_1 g_p^\rho Z_1, \quad C_2 = W_2 g_p^{-\rho} Z_2
\]

where \( \rho \) is a random element from \( \mathbb{Z}_p \). Note that it is not hard to see that \( \text{Game}_{3,|X|} = \text{Game}_4 \).

Suppose there exists an adversary \( A \) that distinguishes between \( \text{Game}_{3,d-1} \) and \( \text{Game}_{3,d} \) with a non-negligible advantage. A simulator \( B \) that solves the C3DH assumption using \( A \) is given: a challenge tuple \( D = (n, \mathbb{G}, \mathbb{G}_T, \sigma, g_p, g_q, g_r, g_p^\alpha, g_p^\beta, g_p^{ab}, g_p^{ab} R_1, g_p^{ab} R_2) \) and \( T \) where \( T = g_p^2 R_3 \) or \( T = R \). Then \( B \) that interacts with \( A \) is described as follows.

Init: \( A \) gives two vectors \( x_0 = (x_{0,1}, \ldots, x_{0,l}), x_1 = (x_{1,1}, \ldots, x_{1,l}) \in \Sigma^l \). \( B \) then flips a random coin \( \gamma \) internally.

Setup: \( B \) first chooses random exponents \( w_1, w_2, \alpha, \beta \in \mathbb{Z}_n, (u_1, h_1'), \ldots, (u_l, h_l') \in \mathbb{Z}_n^2, \) and random elements \( R_v, R_{w_1}, R_{w_2} \in \mathbb{G}_q, (R_{u_1}, R_{h_1}), \ldots, (R_{u_l}, R_{h_l}) \in \mathbb{G}_q^2. \) Next, it publishes a public key as

\[
V = (g_p^{ab} R_1) R_v, \quad W_1 = (g_p^{ab} R_1 \cdot g_p^\alpha) ^{w_1} R_{w_1}, \quad W_2 = g_p^{w_2} R_{w_2}, \quad (U_d = (g_p^\beta) ^{w_2} R_{u,d}, \quad H_d = (g_p^\beta) ^{w_2} R_{h,d}, \quad (U_i = (g_p^\beta) ^{w_2} R_{u,i}, \quad H_i = (g_p^\beta) ^{w_2} R_{h,i}) 1 \leq i \leq d \leq l, \quad \sigma, \quad \Omega = e(g_p^{ab} R_1, g_p^\beta).
\]

Query 1: \( A \) adaptively requests a token for a vector \( \sigma = (\sigma_1, \ldots, \sigma_l) \in \Sigma^l \) to \( B \). Let \( S \) be the set of indexes that are not wild card positions.

Type 1 For Type 1 queries, it is guaranteed that \( d \notin S \) since \( S \cap X = \emptyset \) and \( d \in X \). If \( A \) requests a Type 1 query, then \( B \) chooses random exponents \( r_1', r_2' \in \mathbb{Z}_n \) and random elements \( Y_0, Y_1, Y_2, Y_3 \in \mathbb{G}_r \). Next, it creates a token as

\[
K_0 = g_p^{ab} (g_p^\alpha) ^{w_2' r_1^2} g_p^{w_2' r_2^2} Y_0, \quad K_1 = (g_p^\alpha) ^{w_2' r_1^2} Y_1, \quad K_2 = (g_p^\alpha) ^{-w_2' r_2^2} Y_2, \quad K_3 = g_p^{r_2^2} Y_3.
\]

Note that it is the same as the simulation of the Type 1 token in \( \text{Game}_3 \) if the randomness of the token are defined as

\[
r_1 = w_2' r_1^2 / b \mod p, \quad r_2 = -w_2' r_1^2 / b \mod p, \quad r_3 = r_2' / ab \mod p.
\]

29
Type 2 For Type 2 queries, there exists an index $j \in S$ such that $\sigma_j \neq x_{\gamma,j}$ and there exists two cases such that $d \notin S$ or $d \in S$. Let $\Delta = \sum_{i \in S} u_i^j (\sigma_i - x_{\gamma,i}) \in \mathbb{Z}_P$. Note that $\Delta \neq 0$ except with negligible probability.

In case of $d \notin S$, $B$ chooses random exponents $r_1', r_2', r_3' \in \mathbb{Z}_n$ and random elements $Y_0, Y_1, Y_2, Y_3 \in \mathbb{G}_r$, then it creates a token as

$$K_0 = g_p^{\sigma_j (g_p^a)^{w_1'w_2'r_1'} g_p^{\Delta w_2'r_1'} (g_p^a)^{\sum_{i \in S} h_i'w_2'r_i'} g_p^{\sum_{i \in S} h_i'w_2'r_i'} Y_0,$$

$$K_1 = (g_p^a)^{w_1'r_1'} Y_1, K_2 = (g_p^a)^{-w_1'r_1'} (g_p^b)^{-\Delta w_2'r_2'} Y_2, K_3 = (g_p^a)^{w_2'r_3'} g_p^{w_2'r_3'} Y_3.$$

Note that it is the same as the simulation of the Type 2 token in Game3 if the randomness of the token are defined as

$$r_1 = w_2'r_1'/b \mod p, \quad r_2 = -w_1'r_1'/b - b\Delta r_2'/ab \mod p,$$

$$r_3 = w_2'r_3'/b + w_2'r_2'/ab \mod p.$$

In case of $d \in S$, $B$ chooses random exponents $r_1', r_2', r_3' \in \mathbb{Z}_n$ and random elements $Y_0, Y_1, Y_2, Y_3 \in \mathbb{G}_r$, then it creates a token as

$$K_0 = g_p^{\sigma_j (g_p^a)^{w_1'w_2'r_1'} g_p^{\Delta w_2'r_1'} (g_p^a)^{\sum_{i \in S \setminus \{d\}} h_i'w_2'r_i'} g_p^{\sum_{i \in S \setminus \{d\}} h_i'w_2'r_i'} Y_0,$$

$$K_1 = (g_p^a)^{w_1'r_1'} Y_1, K_2 = (g_p^a)^{-w_1'r_1'} (g_p^b)^{-\Delta w_2'r_2'} Y_2, K_3 = (g_p^a)^{w_2'r_3'} g_p^{w_2'r_3'} Y_3.$$

To show that the above token is the same as the token in Game3, we define the randomness of the token as

$$r_1 = w_2'r_1'/b \mod p, \quad r_2 = -(w_1'r_1'+ h_d'r_3')/b - (b\Delta + h_d')r_2'/ab \mod p,$$

$$r_3 = w_2'r_3'/b + w_2'r_2'/ab \mod p.$$

It is not hard to see that $r_1, r_2, r_3$ are independent random values since $\Delta \neq 0$ except with negligible probability. Therefore, the distribution of the above token is correct as follows

$$K_0 = g_p^{\sigma_j (g_p^a)^{w_1'w_2'} (u_i^{\sigma_i} h_i)^{\tau_3} (\prod_{i \in S \setminus \{d\}} u_i^{\sigma_i} h_i)^{\tau_3} Y_0}$$

$$= g_p^{\sigma_j (g_p^a)^{(ab+1)w_1'} (g_p^b)^{-w_2'(w_1'r_1' + h_d'r_3')/b - (b\Delta + h_d')r_2'/ab} (g_p^{(h_d'(\sigma_d - x_{\gamma,d}) + h_d')} w_2'r_3'/b + w_2'r_2'/ab} Y_0$$

$$= g_p^{\sigma_j (g_p^a)^{w_1'w_2' (u_i^{\sigma_i} h_i)^{\tau_3} (\prod_{i \in S \setminus \{d\}} (abw_2'w_1' + h_i'w_2') Y_0).}$$

**Challenge:** $A$ gives two messages $M_0, M_1$ to $B$. If $M_0 = M_1$, then $B$ computes $C = e(g_p^{abc} R_2, g_p)^{ab} M_T$. Otherwise, it chooses a random elements in $\mathbb{G}_T$ for $C$. Next, it chooses random elements $P, P_{3,1}, \ldots, P_{3,d-1} \in \mathbb{G}_p$ and $Z_0, Z_1, Z_2, Z_{3,1}, \ldots, Z_{3,i} \in \mathbb{G}_q$, then it outputs a challenge ciphertext as

$$C_0 = (g_p^{abc} R_2) Z_0, C_1 = (g_p^{ab} R_2 \cdot P)^{w_1'} Z_1, C_2 = P^{w_2'} Z_2,$$

$$\forall i < d : C_{3,i} = P_{3,i} Z_{3,i}, C_{3,d} = T^{h_d'} Z_{3,d}, \forall i > d : C_{3,i} = (g_p^{abc} R_2)^{h_i} Z_{3,i}.$$ 

If $T$ is a valid C3DH tuple, then $B$ is playing Game3. Otherwise, it is playing Game3.  

**Query 2:** Same as Query Phase 1.

**Guess:** $A$ outputs a guess $\gamma'$. If $\gamma = \gamma'$, it outputs 0. Otherwise, it outputs 1.

This completes our proof.
4.3 HVE in Asymmetric Bilinear Groups

In this section, we construct an HVE scheme in asymmetric bilinear groups of prime order where there are no efficiently computable isomorphisms between two groups $G$ and $\hat{G}$. This construction is algebraically similar to our construction in composite order bilinear groups. In the composite order setting, the subgroups $G_q$ and $G_r$ were used to provide the anonymity of ciphertexts and to hide the correlation between two random values respectively. However, in the prime order asymmetric setting, the non-existence of efficiently computable isomorphisms provides the anonymity of ciphertexts and hides the correlation of two random values in tokens.

4.3.1 Construction

Let $\Sigma = \mathbb{Z}_m$ for some integer $m$ and set $\Sigma_\ast = \mathbb{Z}_m \cup \{\ast\}$. Our scheme is described as follows.

**Setup($1^k$):** The setup algorithm first generates the asymmetric bilinear group $G, \hat{G}$ of prime order $p$ where $p$ is a random prime of bit size $\Theta(\lambda)$ and $p > m$. Let $g, \hat{g}$ be the generators of $G, \hat{G}$ respectively. Next, it chooses random exponents $v', w'_1, w'_2 \in \mathbb{Z}_p$, $(u'_1, h'_1), \ldots, (u'_r, h'_r) \in \mathbb{Z}_p$, and $\alpha, \beta \in \mathbb{Z}_p$. It keeps these as a secret key $SK$ and outputs a public key $PK$ with the description of the asymmetric bilinear group $G, \hat{G}$ as follows

$$
PK = \left( v = g^{v'}, w_1 = g^{w'_1}, w_2 = g^{w'_2}, (u_i = g^{u'_i}, h_i = g^{h'_i})_{i=1}^l, \Omega = e(v, \hat{g})^{\alpha\beta} \right).
$$

**GenToken($\sigma, SK, PK$):** The token generation algorithm takes as input a vector $\sigma = (\sigma_1, \ldots, \sigma_l) \in \Sigma^l_\ast$ and the secret key $SK$. It first selects random exponents $r_1, r_2, r_3 \in \mathbb{Z}_p$ and computes $\hat{v} = g^\hat{v'}, \hat{w}_1 = g^{\hat{w}_1'}, \hat{w}_2 = g^{\hat{w}_2'}, \hat{u}_i = g^{\hat{u}_i'}, \hat{h}_i = g^{\hat{h}_i'}$. Next, it outputs a token as

$$
TK_\sigma = \left( K_0 = g^{\alpha\beta}\hat{u}_i^{\hat{w}_1'}\hat{w}_2' (\prod_{i \in S} \hat{u}_i^{\hat{w}_1'}\hat{h}_i')^{r_1}, K_1 = v^{r_1}, K_2 = \hat{v}^{r_2}, K_3 = \hat{v}^{r_3} \right).
$$

**Encrypt($x, M, PK$):** The encrypt algorithm takes as input a vector $x = (x_1, \ldots, x_l) \in \Sigma^l$, a message $M \in \mathcal{M} \subseteq G_T$ and the public key $PK$. It chooses a random exponent $t \in \mathbb{Z}_p$ and outputs a ciphertext as

$$
CT = \left( C = \Omega^tM, C_0 = v^t, C_1 = w_1^t, C_2 = w_2^t, \{C_{3,i} = (u_i^t h_i)^t \}_{i=1}^l \right).
$$

**Query($CT, TK_\sigma, PK$):** The query algorithm takes as input a ciphertext $CT$ and a token $TK_\sigma$ with a vector $\sigma$. It first computes

$$
M \leftarrow C \cdot e(C_0, K_0)^{-1} \cdot e(C_1, K_1) \cdot e(C_2, K_2) \cdot e(\prod_{i \in S} C_{3,i}, K_3).
$$

If $M \notin \mathcal{M}$, it outputs $\bot$ indicating that the predicate $f_\sigma$ is not satisfied. Otherwise, it outputs $M$ indicating that the predicate $f_\sigma$ is satisfied.

**Remark 4.** We can expand the finite space $\Sigma$ from $\mathbb{Z}_m$ to all of $\{0,1\}^3$ by using a collision-resistant hash function for the vector of attributes.
4.3.2 Security

**Theorem 4.3.1.** The above HVE construction is selectively secure under the decisional aBDH assumption, the decisional aDH assumption, and the decisional a3DH assumption.

**Proof.** The main structure of this proof is almost the same as the proof of Theorem 4.2.1. That is, it consists of a sequence of Game$_0$, Game$_1$, Game$_2$, Game$_3$, Game$_4$ games, and we prove that there is no probabilistic polynomial-time adversary that distinguishes between Game$_{i-1}$ and Game$_i$. These games are nearly the same as those in the proof of Theorem 4.2.1. The difference is that the ciphertext elements and the token elements are represented in prime order groups, whereas those elements were represented in composite order groups in the proof of Theorem 4.2.1. The main components of the decisional aBDH assumption under prime order asymmetric bilinear groups are the same as the decisional cBDH assumption. Note that the BSD assumption for Theorem 4.2.1 is not needed. Second, the indistinguishability between Game$_1$ and Game$_2$ can be proven using the decisional aDH assumption for $\hat{\mathbb{G}}$ under prime order asymmetric bilinear groups. The proof is the same as Lemma 4.2.3, since the decisional C2DH assumption in Lemma 4.2.3 is converted to the decisional aDH assumption in prime order asymmetric bilinear groups. Finally, the indistinguishability between Game$_2$ and Game$_3$, (the indistinguishability between Game$_3$ and Game$_4$, respectively) can be proven under the decisional a3DH assumption. The proof is the same as Lemma 4.2.4 (Lemma 4.2.5 respectively) except using the decisional a3DH instead of the decisional C3DH assumption, since the decisional C3DH assumption can be converted to the decisional a3DH in prime order asymmetric bilinear groups. This completes our proof.

4.3.3 Discussion

4.3.4 Freeman Method

Recently, a heuristic methodology that converts cryptosystems from composite order bilinear groups to prime order asymmetric bilinear groups was proposed by Freeman in [16]. The main idea of Freeman’s method is constructing a product group $\mathbb{G}^n$ that has orthogonal subgroups by applying the direct product to a prime order bilinear group $\mathbb{G}$ where $n$ is the number of subgroups. Our construction in composite order bilinear groups is also converted to a new construction in prime order asymmetric bilinear groups by applying Freeman’s method. However, the new construction requires three group elements of the prime order group to represent one element in the composite order group since Freeman’s method converts one element of composite order groups with three subgroups to three elements of prime order groups. That is, the number of groups elements in ciphertexts and tokens, and the number of pairing operations in decryption increase by three times.
Chapter 5

Convert HVE from Composite to Prime Order Groups

5.1 Overview

In this chapter, we construct HVE schemes that are secure under any kind of pairing types and prove their selective model security. To achieve our goals, we first present a framework that converts HVE schemes that are the extreme generalization of AIBE from composite order bilinear groups to prime order bilinear groups.

The previous conversion methods that convert cryptographic schemes from composite order bilinear groups to prime order bilinear groups are Freeman’s method and Ducas’ method [15, 16]. The conversion method of Ducas is that random blinding elements in ciphertexts can be eliminated in asymmetric bilinear groups of prime order since the decisional Diffie-Hellman (DDH) assumption holds in asymmetric bilinear groups. Using this method, Ducas converted some anonymous hierarchical IBE (AHIBE) and HVE schemes from bilinear groups of composite order to asymmetric bilinear groups of prime order. The conversion method of Freeman is that product groups using a direct product of groups and vector orthogonality using an inner product operation provide the subgroup decision hardness and the subgroup orthogonality properties in prime order bilinear groups, respectively. The merit of Freeman’s method is that it convert almost all cryptographic schemes from bilinear groups of composite order to asymmetric bilinear groups of prime order. The demerits of Freeman’s method are that the resulting schemes work in asymmetric bilinear groups and use complex assumptions that depend on complex basis vectors.

The conversion method of this paper is similar to the conversion method of Freeman in terms of using product groups and vector orthogonality, but it has the following three differences. The first difference is that Freeman’s method is related to the subgroup decision (SGD) assumption in prime order bilinear groups, whereas our method is not related to the SGD assumption. The second difference is that Freeman’s method only works in asymmetric bilinear groups of prime order, whereas our method works in any bilinear groups of prime order. The third difference is that the cryptographic schemes from Freeman’s method use complex assumptions that depend on complex basis vectors, whereas the HVE schemes from our method use simple assumptions that are independent of basis vectors.

We first convert the HVE scheme of Boneh and Waters, the delegatable HVE scheme of Shi and Waters, and the efficient HVE scheme with constant cost of pairing of Lee and Lee from composite order bilinear groups to prime order bilinear groups. Next we prove that these converted HVE schemes are selectively secure under the decisional Bilinear Diffie-Hellman (BDH) and the decisional Parallel 3-party Diffie-Hellman
Definition 5.2.1 (Vector Operations). Let $G$ be multiplicative cyclic groups of prime $p$ order. Let $g$ be a generator of $G$. We define vector operations over $G$ as follows:

1. For a vector $b = (b_1, \ldots, b_n) \in \mathbb{Z}_p^n$, define $g^b := (g^{b_1}, \ldots, g^{b_n}) \in \mathbb{G}^n$.

2. For a vector $b = (b_1, \ldots, b_n) \in \mathbb{Z}_p^n$ and a scalar $c \in \mathbb{Z}_p$, define $(g^b)^c := (g^{b_1c}, \ldots, g^{b_nc}) \in \mathbb{G}^n$.

3. For two vectors $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{Z}_p^n$, define $g^a g^b := (g^{a_1+b_1}, \ldots, g^{a_n+b_n}) \in \mathbb{G}^n$.

Definition 5.2.2 (Bilinear Product Groups). Let $(p, \mathbb{G}, \mathbb{G}_T, e)$ be bilinear groups of prime order. Let $g$ be a generator of $G$. For integers $n$ and $m$, the bilinear product groups $((p, \mathbb{G}, \mathbb{G}_T, e), g^{b_1}, \ldots, g^{b_m})$ of basis vectors $b_1, \ldots, b_m$ is defined as follows

\begin{table}[h]
\centering
\caption{Comparison between previous HVE schemes and ours}
\begin{tabular}{|c|c|c|c|c|}
\hline
Scheme & Group Order & Pairing Type & Ciphertext Size & No. of Pairing \\
\hline
BW-HVE \cite{13} & $p_1 p_2$ & Type 1 & $(2l + 1) |G| + |G_T|$ & $2s + 1$ \\
KSW-IPE \cite{24} & $p_1 p_2 p_3$ & Type 1 & $(4l + 1) |G| + |G_T|$ & $4l + 1$ \\
SW-dHVE \cite{36} & $p_1 p_2 p_3$ & Type 1 & $(l + 3) |G_1| + |G_T|$ & $s + 3$ \\
OT-IPE \cite{29} & $p$ & Type 1,2,3 & $(2l + 3) |G| + |G_T|$ & $2l + 3$ \\
Duc-dHVE \cite{15} & $p$ & Type 3 & $(l + 3) |G_1| + |G_T|$ & $s + 3$ \\
Par-IPE \cite{31} & $p$ & Type 1,2,3 & $(8l + 2) |G| + |G_T|$ & $8s + 2$ \\
LL-HVE \cite{26} & $p_1 p_2 p_3$ & Type 1 & $(l + 3) |G| + |G_T|$ & $4$ \\
Ours (BW-HVE) & $p$ & Type 1,2,3 & $(l + 3) |G_1| + |G_T|$ & $4$ \\
Ours (SW-dHVE) & $p$ & Type 1,2,3 & $(3l + 9) |G| + |G_T|$ & $3s + 9$ \\
Ours (LL-HVE) & $p$ & Type 1,2,3 & $(3l + 9) |G_1| + |G_T|$ & $12$ \\
\hline
\end{tabular}
\end{table}

\(p = \text{prime value, } l = \text{no. of attributes in ciphertext, } s = \text{no. of attributes in token}\)

(P3DH) assumptions. Through this conversion, we constructed the first delegatable HVE scheme and efficient HVE scheme with constant cost of pairing in any bilinear groups of prime order. The previous HVE schemes and ours are compared in Table 5.1. In Table 5.1, HVE schemes from IPE schemes are also included since IPE imply HVE. Finally, we prove that the new decisional P3DH assumption is secure in generic group model that was introduced by Shoup in \cite{37}.

5.2 Framework

The basic idea to convert HVE schemes from composite order bilinear groups to prime order bilinear groups is to use bilinear product groups that are extended from bilinear groups using the direct product operation. Bilinear product groups were widely used in dual system encryption of Waters \cite{28,40}, private linear broadcast encryption of Garg et al. \cite{18}, and the conversion method of Freeman \cite{16}. The product groups extended from multiplicative cyclic groups represent an exponent as a vector. Thus vector operations in product groups should be defined. Since bilinear groups have bilinear maps, the bilinear maps on bilinear product groups should be defined. Definition 5.2.1 and Definition 5.2.2 define the vector operations in product groups and bilinear product groups, respectively.

\(= \text{no. of attributes in ciphertext, } s = \text{no. of attributes in token}\)
1. The basis vectors $\mathbf{b}_1, \ldots, \mathbf{b}_m$ are random vectors such that $\mathbf{b}_i = (b_{i,1}, \ldots, b_{i,n}) \in \mathbb{Z}_p^n$.

2. The bilinear map $e : \mathbb{G}^n \times \mathbb{G}^n \to \mathbb{G}_T$ is defined as $e(g^a, g^b) := \prod_{i=1}^n \hat{e}(g^{a_i}, g^{b_i}) = \hat{e}(g, g)^{a \cdot b}$ where $\cdot$ is the inner product operation.

To guarantee the correctness of cryptographic schemes in bilinear product groups, the orthogonal property of composite order bilinear groups should be implemented in bilinear product groups. The previous researches \cite{16 18 28 40} showed that the orthogonal property can be implemented in bilinear product groups. The idea is that the orthogonality between vectors can be defined using the inner-product operation such that $x \cdot y = 0$ since the bilinear map provides the inner-product operation. Definition 5.2.3 define the orthogonality in bilinear product groups.

**Definition 5.2.3 (Orthogonality).** Let $(\langle p, \mathbb{G}, \mathbb{G}_T, e \rangle, g^{b_1}, \ldots, g^{b_m})$ be bilinear product groups with $n, m$ parameters. Let $G_i, G_j$ be subgroups spanned by $g^{b_i}, g^{b_j}$, respectively. That is, $G_i = \langle g^{b_i} \rangle$ and $G_j = \langle g^{b_j} \rangle$. Then the two subgroups $G_i$ and $G_j$ are orthogonal to each other if $e(A, B) = 1$ for all $A \in G_i$ and $B \in G_j$.

The main idea of our method that convert HVE schemes from composite order bilinear groups to prime order bilinear groups is the previous HVE schemes \cite{13 26 36} in composite order bilinear groups use the decisional Composite 3-party Diffie-Hellman (C3DH) assumption that is not a kind of the subgroup decision (SGD) assumption.

The SGD assumption is to distinguish whether $h \in \mathbb{G}$ or $h \in \mathbb{G}_1$ where $\mathbb{G}$ is a group and $\mathbb{G}_1$ is a subgroup of $\mathbb{G}$ \cite{10}. In product groups $\mathbb{G}^n$, a subgroup $G$ is defined as a vector space spanned by some basis vectors $\mathbf{b}_1, \ldots, \mathbf{b}_m$ such that $G = \langle g^{b_1}, \ldots, g^{b_m} \rangle$. If a subgroup is constructed from one basis vector, then the SGD assumption is related to the DDH assumption. If a subgroup is constructed from $k$ number of basis vectors, then the SGD assumption is related to the decisional $k$-Linear ($k$-DLIN) assumption \cite{16}. In symmetric bilinear groups of prime order, a subgroup should be constructed from two basis vectors since the DDH assumption is not valid \cite{18 40}. If a subgroup is constructed from two basis vectors, then cryptographic schemes become complicated and there is no generic conversion method from composite order groups to prime order groups. In asymmetric bilinear groups of prime order, a subgroup can be constructed from one basis vector since the DDH assumption is valid \cite{16 28}. If a subgroup is constructed from one basis vector, then there is a generic conversion method of Freeman, but it only works in asymmetric bilinear groups.

The decisional C3DH assumption is defined in Definition 5.2.4. The notable properties of the decisional C3DH assumption are that the $T$ value is always an element of $\mathbb{G}_{p_1 p_2}$ in contrast to the SGD assumption, and the subgroup $\mathbb{G}_{p_3}$ plays the role of random blinding. From these properties of the C3DH assumption, it is possible to use just one basis vector to construct a subgroup. Additionally, it is possible to use simple basis vectors for cryptographic schemes since ciphertexts and tokens can use different subgroups that are not orthogonal.

**Definition 5.2.4 (Composite 3-party Diffie-Hellman (C3DH) Assumption).** Let $(n, \mathbb{G}, \mathbb{G}_T, e)$ be a description of bilinear groups of composite order $n = p_1 \cdots p_m$ where $p_i$ is a random prime. Let $g_{p_i}$ be a generator of the subgroup $\mathbb{G}_{p_i}$. The decisional C3DH assumption is stated as follows: given a challenge tuple $D = (n, \mathbb{G}, \mathbb{G}_T, e, g_{p_1}, \ldots, g_{p_m}, \hat{g}_{p_1}, \hat{g}_{p_1}, \hat{g}_{p_1}, \hat{g}_{p_1}, \hat{g}_{p_1}, \hat{g}_{p_1}, R_1, \hat{R}_1, R_2, \hat{R}_2, R_3, \hat{R}_3) \in \mathbb{G}_{p_3}$ and $T$, decides whether $T = T_0 = g_{p_1} R_3$ or $T = T_1 = \hat{g}_{p_1} R_3$ with random choices of $a, b, c, d \in \mathbb{Z}_{p_1}$ and $R_1, R_2, R_3 \in \mathbb{G}_{p_2}$.

For instance, we select basis vectors $\mathbf{b}_{1,1} = (1, 0), \mathbf{b}_{1,2} = (1, a), \mathbf{b}_2 = (a, -1)$ for the conversion from bilinear groups of composite order $n = p_1 p_2$ order. For the conversion from bilinear groups of composite order $n = p_1 p_2 p_3$ order, we select basis vectors $\mathbf{b}_{1,1} = (1, 0, a_1), \mathbf{b}_{1,2} = (1, a_2, 0), \mathbf{b}_2 = (a_2, -1, a_1 a_2 - a_3), \mathbf{b}_3 = (a_1, a_3, -1)$. Though different basis vectors for the structure of composite order were selected, the assumption for the security proof is the simple one that is independent of basis vectors.
5.3 Conversion 1: BW-HVE

In this section, we convert the HVE scheme of Boneh and Waters \[13\] to prime order bilinear groups and prove its selective model security under the decisional BDH and P3DH assumptions.

5.3.1 Construction

Setup(1\(^\lambda\), I): The setup algorithm first generates the bilinear group \( G \) of prime order \( p \) of bit size \( \Theta(\lambda) \). It chooses a random value \( a \in \mathbb{Z}_p \) and sets basis vectors for bilinear product groups as \( b_{1,1} = (1, 0) \), \( b_{1,2} = (1, a) \), \( b_2 = (a, -1) \). Next, it chooses random exponents \( \nu', \{u'_i, h'_i, w'_i\}_{i=1}^l \), \( \alpha \in \mathbb{Z}_p \), and it computes the following values using the basis vectors

\[
\begin{align*}
B_{1,1} &= g^{b_{1,1}}, \quad B_{1,2} = g^{b_{1,2}}, \quad B_2 = g^{b_2}, \\
\mathbf{g}^{\nu'_i} &= B_{1,1}^{\nu'_i}, \quad \{g^{u'_i} = B_{1,1}^{u'_i}, g^{h'_i} = B_{1,1}^{h'_i}, g^{w'_i} = B_{1,1}^{w'_i}\}_{i=1}^l, \\
\mathbf{g}^{\nu''_i} &= B_{1,2}^{\nu''_i}, \quad \{g^{u''_i} = B_{1,2}^{u''_i}, g^{h''_i} = B_{1,2}^{h''_i}, g^{w''_i} = B_{1,2}^{w''_i}\}_{i=1}^l.
\end{align*}
\]

It keeps \( \mathbf{g}^{\nu''}, \{g^{u''_i}g^{h''_i}g^{w''_i}\}_{i=1}^l \), \( (g^{b_{1,2}})^{\alpha} \) as a secret key \( \mathbf{SK} \). Then it publishes a public key \( PK \) using random blinding values \( \{z_{a,i}, z_{h,i}, z_{w,i}\}_{i=1}^l \in \mathbb{Z}_p \) as follows

\[
PK = \left( B_{1,1}, B_{1,2}, B_2, \mathbf{V} = g^{\nu'_1}B_2^{\nu''_1}, \{U_i = g^{u'_i}B_2^{\nu''_i}, H_i = g^{h'_i}B_2^{\nu''_i}, W_i = g^{w'_i}B_2^{\nu''_i}\}_{i=1}^l, \right.
\]

\[
\left. \Omega = e(\mathbf{g}^{\nu'_1}, g^{b_{1,2}})^{\alpha} \right).
\]

GenToken(\(\sigma, \mathbf{SK}, PK\)): The token generation algorithm takes as input an attribute vector \( \sigma = (\sigma_1, \ldots, \sigma_l) \in \Sigma_l \) and the secret key \( \mathbf{SK} \). Let \( S \) be the set of indexes that are not wild-card fields in the vector \( \sigma \). It selects random exponents \( \{r_1,i, r_2,i\}_{i\in S} \in \mathbb{Z}_p \) and outputs a token as

\[
\mathbf{TK}_\sigma = \left( K_1 = (g^{b_{1,2}})^{\alpha} \prod_{i\in S} (g^{u''_i})^{r_1,i}(g^{w''_i})^{r_2,i}, \{K_{2,i} = (g^{\nu''_i})^{-r_1,i}, K_{3,i} = (g^{\nu''_i})^{-r_2,i}\}_{i\in S} \right).
\]

Encrypt(\(x, M, PK\)): The encryption algorithm takes as input an attribute vector \( x = (x_1, \ldots, x_l) \in \Sigma^l \), a message \( M \in \mathcal{M} \subseteq \mathbb{G}_T \), and the public key \( PK \). It first chooses a random exponent \( t \in \mathbb{Z}_p \) and random blinding values \( z_1, \{z_{2,i}, z_{3,i}\}_{i=1}^l \in \mathbb{Z}_p \). Then it outputs a ciphertext as

\[
\mathbf{CT} = \left( C_0 = \Omega' M, C_1 = V^t B_2^{z_1}, \{C_{2,i} = (U_i H_i)^t B_2^{z_{2,i}}, C_{3,i} = W_i B_2^{z_{3,i}}\}_{i=1}^l \right).
\]

Query(\(\mathbf{CT}, \mathbf{TK}_\sigma, PK\)): The query algorithm takes as input a ciphertext \( \mathbf{CT} \) and a token \( \mathbf{TK}_\sigma \) of a vector \( \sigma \). It first computes

\[
M \leftarrow C_0 \cdot \left( e(C_1, K_1) \cdot \prod_{i\in S} (e(C_{2,i}K_{2,i}) \cdot e(C_{3,i}K_{3,i})) \right)^{-1}.
\]

If \( M \notin \mathcal{M} \), it outputs \( \bot \) indicating that the predicate \( f_\sigma \) is not satisfied. Otherwise, it outputs \( M \) indicating that the predicate \( f_\sigma \) is satisfied.
5.3.2 Correctness

If \( f_σ(x) = 1 \), then the following calculations shows that \( \text{Query}(\text{CT}, \text{TK}_σ, \text{PK}) = M \) using the orthogonality of basis vectors such that \( e(g^{b_1}, g^{b_1^2}) = 1 \).

\[
e(C_1, K_1) \cdot \prod_{i \in S} \left( e(C_{2,i}, K_{2,i}) \cdot e(C_{3,i}, K_{3,i}) \right)
= e(g^{y_i^1}, g^{b_1^2}) \cdot \prod_{i \in S} \left( e(g^{u_{i1}}^1 g^{b_1^2}, (g^{y_i^2})^{r_{1,i}} \cdot (g^{y_i^2})^{r_{2,i}}) \cdot e((g^{w_i^1})^{r_{1,i}}, (g^{w_i^2})^{r_{2,i}}) \right)
= e(g^{y_i^1}, g^{b_1^2}) \cdot \prod_{i \in S} e(g^{v_i}, (g^{u_i^1})^{r_{1,i}} \cdot (g^{v_i})^{r_{2,i}} = e(g^{y_i^1}, g^{b_1^2})^{aT}.
\]

Otherwise, that is \( f_σ(x) = 0 \), then the probability of \( \text{Query}(\text{CT}, \text{TK}_σ, \text{PK}) \neq \perp \) is negligible by limiting \(|M| \) to less than \(|G_T|^{1/4} \).

5.3.3 Security

Theorem 5.3.1. The above HVE construction is selectively secure under the decisional BDH and P3DH assumptions.

The proof of this theorem is easily obtained from the following four Lemma 5.3.2 and Assumption 5.3.3, 5.3.4, and 5.3.5. Before presenting the four lemmas, we first introduce the following three assumptions. The HVE scheme of Boneh and Waters constructed in bilinear groups of composite \( n = p_1 p_2 \) order, and its security was proven under the decisional BDH, Bilinear Subgroup Decision (BSD), and the decisional C3DH assumptions [13]. These assumptions in composite order bilinear groups are converted to the following Assumptions 5.3.1, 5.3.2, and 5.3.3 using our conversion method.

Assumption 5.3.1 Let \((p, G, G_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_{2}}\) be the bilinear product group of basis vectors \( b_{1,1} = (1,0), b_{1,2} = (1,a), \) and \( b_2 = (a, -1) \). The Assumption 5.3.1 is stated as follows: given a challenge tuple

\[
D = \left((p, G, G_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_{2}}, (g^{b_{1,1}})^{c_1}, (g^{b_{1,2}})^{c_2}, (g^{b_{2}})^{c_3}\right) \text{ and } T,
\]
decides whether \( T = T_0 = e(g,g)^{c_1 c_2 c_3} \) or \( T = T_1 = e(g,g)^d \) with random choices of \( c_1, c_2, c_3, d \in \mathbb{Z}_p \).

Assumption 5.3.2 Let \((p, G, G_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_{2}}\) be the bilinear product group of basis vectors \( b_{1,1} = (1,0), b_{1,2} = (1,a), \) and \( b_2 = (a, -1) \). The Assumption 5.3.2 is stated as follows: given a challenge tuple

\[
D = \left((p, G, G_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_{2}}\right) \text{ and } T,
\]
decides whether \( T = T_0 = e((g^{b_{1,1}})^{c_1}, (g^{b_{2}})^{c_2}) \) or \( T = T_1 = e((g^{b_{1,1}})^{c_1}, (g^{b_{2}})^{c_2}) \) with random choices of \( c_1, c_2, c_3 \in \mathbb{Z}_p \).

Assumption 5.3.3 Let \((p, G, G_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_{2}}\) be the bilinear product group of basis vectors \( b_{1,1} = (1,0), b_{1,2} = (1,a), \) and \( b_2 = (a, -1) \). The Assumption 5.3.3 is stated as follows: given a challenge tuple

\[
D = \left((p, G, G_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_{2}}, (g^{b_{1,1}})^{c_1}, (g^{b_{1,2}})^{c_2}, (g^{b_{2}})^{c_3}\right) \text{ and } T,
\]
decides whether \( T = T_0 = (g^{b_{1,1}})^{c_1} (g^{b_{2}})^{c_3} \) or \( T = T_1 = (g^{b_{1,1}})^d (g^{b_{2}})^{c_2} \) with random choices of \( c_1, c_2, c_3, d \in \mathbb{Z}_p \) and \( z_1, z_2, z_3 \in \mathbb{Z}_p \).

Lemma 5.3.2. The above HVE construction is selectively secure under the Assumptions 5.3.1, 5.3.2, and 5.3.3.
Proof. The proof of this lemma is directly obtained from [13] since the Assumptions 5.3-1, 5.3-2, and 5.3-2 in prime order bilinear groups are correspond to the Bilinear Diffie-Hellman (BDH), Bilinear Subgroup Decision (BSD), and Composite 3-party Diffie-Hellman (C3DH) assumptions in composite order bilinear groups. That is, the proof of [13] can be exactly simulated using the vector operations in the Definition 5.2.4 and the Assumptions 5.3-1, 5.3-2, and 5.3-3.

Lemma 5.3.3. If the decisional BDH assumption holds, then the Assumption 5.3-1 also holds.

Proof. Suppose there exists an adversary \( \mathcal{A} \) that breaks the Assumption 5.3-1 with a non-negligible advantage. An algorithm \( \mathcal{B} \) that solves the decisional BDH assumption using \( \mathcal{A} \) is given: a challenge tuple \( D = ((p, \mathbb{G}, \mathbb{G}_T, e), g, g^{c_1}, g^{c_2}, g^{c_3}) \) and \( T \) where \( T = T_0 = e(g, g)^{c_1 c_2 c_3} \) or \( T = T_1 = e(g, g)^d \). \( \mathcal{B} \) first chooses random values \( a \in \mathbb{Z}_p \) and computes

\[
\begin{align*}
g^{b_{1,1}} &= (g, 1), \quad g^{b_{1,2}} = (g, g^a), \quad g^{b_2} = (g^a, g^{-1}), \\
(g^{b_{1,1}})^{c_1} &= (g^{c_1}, 1), \quad (g^{b_{1,2}})^{c_2} = (g^{c_2}, 1), \quad (g^{b_{1,1}})^{c_3} = (g^{c_3}, 1), \\
(g^{b_{1,2}})^{c_1} &= (g^{c_1}, (g^{c_1})^a), \quad (g^{b_{1,2}})^{c_2} = (g^{c_2}, (g^{c_2})^a).
\end{align*}
\]

Next, it gives the tuple \( D' = ((p, \mathbb{G}, \mathbb{G}_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_2}, (g^{b_{1,1}})^{c_1}, (g^{b_{1,2}})^{c_2}, (g^{b_{1,1}})^{c_3}, (g^{b_{1,2}})^{c_3}, (g^{b_{1,1}})^{c_2}, (g^{b_{1,2}})^{c_1}) \) and \( T \) to \( \mathcal{A} \). Then \( \mathcal{A} \) outputs a guess \( \gamma' \). \( \mathcal{B} \) also outputs \( \gamma' \). If the advantage of \( \mathcal{A} \) is \( \varepsilon \), then the advantage of \( \mathcal{B} \) is greater than \( \varepsilon \) since the distribution of the challenge tuple to \( \mathcal{A} \) is equal to the Assumption 5.3-1.

Lemma 5.3.4. The Assumption 5.3-2 holds for all adversaries.

Proof. The equation \( e((g^{b_{1,1}})^{c_1}, (g^{b_{1,2}})^{c_2}, (g^{b_{1,2}})^{c_2}) = e((g^{b_{1,1}})^{c_1}, (g^{b_{1,2}})^{c_2}) \) holds by the orthogonality of basis vectors such that \( e(g^{b_2}, g^{b_{1,2}}) = 1 \). Therefore, any adversary can not break the Assumption 5.3-2.

Lemma 5.3.5. If the decisional P3DH assumption holds, then the Assumption 5.3-3 also holds.

Proof. Suppose there exists an adversary \( \mathcal{A} \) that breaks the Assumption 5.3-3 with a non-negligible advantage. An algorithm \( \mathcal{B} \) that solves the decisional P3DH assumption using \( \mathcal{A} \) is given: a challenge tuple \( D = ((p, \mathbb{G}, \mathbb{G}_T, e), (g, f), (g^{c_1}, f^{c_1}), (g^{c_2}, f^{c_2}), (g^{c_1 c_2}, f^{c_1}, g^{c_2}), (g^{c_1 c_2}, f^{c_2}, g^{c_2})) \) and \( T \) where \( T = T_0 = (g^{c_2} f^{c_2}, g^{c_3}) \) or \( T = T_1 = (g^d f^{c_2}, g^{c_3}) \). \( \mathcal{B} \) first computes

\[
\begin{align*}
g^{b_{1,1}} &= (g, 1), \quad g^{b_{1,2}} = (g, f), \quad g^{b_2} = (f, g^{-1}), \\
(g^{b_{1,1}})^{c_1} &= (g^{c_1}, f^{c_1}), \quad (g^{b_{1,2}})^{c_2} = (g^{c_2}, f^{c_2}), \\
(g^{b_{1,1}})^{c_1} c_2 g^{b_{1,2}} c_2 &= (g^{c_1 c_2}, f^{c_1}, (g^{c_2})^{-1}).
\end{align*}
\]

Intuitively, it sets \( c = d \log(f) \). Next, it gives the tuple \( D' = ((p, \mathbb{G}, \mathbb{G}_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_2}, (g^{b_{1,1}})^{c_1}, (g^{b_{1,2}})^{c_2}, (g^{b_{1,1}})^{c_2}, (g^{b_{1,2}})^{c_1}, (g^{b_{1,2}})^{c_2}, (g^{b_{1,1}})^{c_1 c_2 c_3}) \) and \( T \) to \( \mathcal{A} \). Then \( \mathcal{A} \) outputs a guess \( \gamma' \). \( \mathcal{B} \) also outputs \( \gamma' \). If the advantage of \( \mathcal{A} \) is \( \varepsilon \), then the advantage of \( \mathcal{B} \) is greater than \( \varepsilon \) since the distribution of the challenge tuple to \( \mathcal{A} \) is equal to the Assumption 5.3-3.

5.4 Conversion 2: LL-HVE

In this section, we convert the HVE scheme of Lee and Lee [26] to prime order bilinear groups and prove its selective model security under the decisional BDH and P3DH assumptions.
5.4.1 Construction

Setup($\mathbb{G}$): The setup algorithm first generates the bilinear group $\mathbb{G}$ of prime order $p$ of bit size $\Theta(\lambda)$. It chooses random values $a_1, a_2, a_3 \in \mathbb{Z}_p$ and sets basis vectors for bilinear product groups as $b_{1,1} = (1, 0, a_1)$, $b_{1,2} = (1, a_2, 0)$, $b_2 = (a_2, -1, a_2 - a_3)$, $b_3 = (a_1, a_3, -1)$. Next, it chooses random exponents $v', w'_1, w'_2, \{u'_i, h_i\}_{i=1}^l, \alpha \in \mathbb{Z}_p$, and it computes the following values using the basis vectors

\[
B_{1,1} = g^{b_{1,1}}, \quad B_{1,2} = g^{b_{1,2}}, \quad B_2 = g^{b_2}, \quad B_3 = g^{b_3},
\]

\[
g^{v_1} = B_{1,1}^{v_1}, \quad g^{w_{1,1}} = B_{1,1}^{w_{1,1}}, \quad g^{w_{1,2}} = B_{1,1}^{w_{1,2}}, \quad \{g^{u_i} = B_{1,1}^{u_i}, \quad g^{h_i} = B_{1,1}^{h_i}\}_{i=1}^l,
\]

\[
g^{v_2} = B_{1,2}^{v_2}, \quad g^{w_{1,1}} = B_{1,2}^{w_{1,1}}, \quad g^{w_{2,2}} = B_{1,2}^{w_{2,2}}, \quad \{g^{u_i} = B_{1,2}^{u_i}, \quad g^{h_i} = B_{1,2}^{h_i}\}_{i=1}^l.
\]

It keeps $g^{v_2}, g^{w_{1,1}}, g^{w_{2,2}}, \{g^{u_i}, g^{h_i}\}_{i=1}^l, (g^{b_{1,2}})^{\alpha}$ as a secret key SK. Then it publishes a public key PK using random blinding values $z_0, z_{w,1}, z_{w,2}, \{z_{u,i}, z_{h,i}\}_{i=1}^l \in \mathbb{Z}_p$ as follows

\[
PK = \left( \begin{array}{c}
B_{1,1}, B_{1,2}, B_2, B_3, V = g^{v_1}B_{2}^{v_2}, W_1 = g^{w_{1,1}}B_{2}^{w_{1,2}}, W_2 = g^{w_{2,2}}B_{2}^{w_{2,1}}, \\
\{U_i = g^{u_i}B_{2}^{u_i}, H_i = g^{h_i}B_{2}^{h_i}\}_{i=1}^l, \quad \Omega = e(g^{v_1}, g^{b_{1,2}})^{\alpha}\end{array} \right).
\]

GenToken($\sigma, SK, PK$): The token generation algorithm takes as input an attribute vector $\sigma = (\sigma_1, \ldots, \sigma_l) \in \Sigma$ and the secret key $SK$. Let $S$ be the set of indexes that are not wild-card fields in the vector $\bar{\sigma}$. It selects random exponents $r_1, r_2, r_3 \in \mathbb{Z}_p$ and random blinding values $y_1, y_2, y_3, y_4 \in \mathbb{Z}_p$. Next it outputs a token as

\[
TK_{\bar{\sigma}} = \left( \begin{array}{c}
K_1 = (g^{b_{1,2}})^{\alpha} (g^{w_{1,2}})^{r_1} (g^{w_{2,2}})^{r_2} \left( \prod_{i \in S} (g^{u_i})^{\sigma_i} g^{h_i} \right)^{r_3} B_{3}^{y_1}, \\
K_2 = (g^{v_2})^{-r_1} B_{3}^{y_2}, \quad K_3 = (g^{v_2})^{-r_2} B_{3}^{y_3}, \quad K_4 = (g^{v_2})^{-r_3} B_{3}^{y_4} \end{array} \right).
\]

Encrypt($x, M, PK$): The encryption algorithm takes as input an attribute vector $x = (x_1, \ldots, x_l) \in \Sigma'$, a message $M \in \mathcal{M} \subseteq \mathbb{G}_T$, and the public key $PK$. It first chooses a random exponent $t \in \mathbb{Z}_p$ and random blinding values $z_1, z_2, z_3, \{z_{d,i}\}_{i=1}^l \in \mathbb{Z}_p$. Then it outputs a ciphertext as

\[
CT = \left( \begin{array}{c}
C_0 = \Omega' M, \quad C_1 = V^t B_{2}^{y_1}, \quad C_2 = W_{1}^t B_{2}^{y_2}, \quad C_3 = W_{2}^t B_{2}^{y_3}, \quad \{C_{4,i} = (U_i V H_i)^t B_{2}^{y_2}\}_{i=1}^l \end{array} \right).
\]

Query($CT, TK_{\bar{\sigma}}, PK$): The query algorithm takes as input a ciphertext $CT$ and a token $TK_{\bar{\sigma}}$ of a vector $\bar{\sigma}$. It first computes

\[
M \leftarrow C_0 \cdot e(C_1, K_1) \cdot e(C_2, K_2) \cdot e(C_3, K_3) \cdot e(\prod_{i \in S} C_{4,i}, K_4)^{-1}.
\]

If $M \notin \mathcal{M}$, it outputs $\bot$ indicating that the predicate $f_{\bar{\sigma}}$ is not satisfied. Otherwise, it outputs $M$ indicating that the predicate $f_{\bar{\sigma}}$ is satisfied.
5.4.2 Correctness

If \( f_\theta(x) = 1 \), then the following calculation shows that \( \text{Query}(\mathcal{CT}, TK_\theta, PK) = M \) by the orthogonality of basis vectors such that \( e(g^{b_{1,1}}, g^{b_1}) = 1, e(g^{b_{1,2}}, g^{b_2}) = 1, e(g^{b_2}, g^{b_1}) = 1 \).

\[
e(\mathbf{C}_1, \mathbf{K}_1) \cdot e(\mathbf{C}_2, \mathbf{K}_2) \cdot e(\mathbf{C}_3, \mathbf{K}_3) \cdot e\left(\prod_{i \in S} \mathbf{C}_4, \mathbf{K}_4\right)
\]
\[
= e\left(\mathbf{g}_{w_1}^{i'}, (g^{b_{1,2}})^{a_i} (g^{w_{2,2}})^{\alpha_i} \prod_{i \in S} (g^{u_i} (g^{w_{1,2}})^{\alpha_i})^{\tau_i}\right).
\]
\[
e\left(\mathbf{g}_{w_1}^{i'}, (g^{w_{2,2}})^{-\tau_i}\right) \cdot e\left(\mathbf{g}_{w_2}^{i'}, (g^{w_{2,2}})^{-\tau_i}\right) \cdot e\left(\prod_{i \in S} (g^{u_i} (g^{w_{2,2}})^{\alpha_i})^{\tau_i}, (g^{v_i})^{-\tau_i}\right)
\]
\[
= e\left(\mathbf{g}_{w_1}^{i'}, (g^{b_{1,2}})^{a_i} \prod_{i \in S} (g^{u_i} (g^{w_{1,2}})^{\alpha_i})^{\tau_i}\right) = e(g^{v_i}, g^{b_{1,2}})^{a_i}.
\]

Otherwise, that is \( f_\theta(x) = 0 \), the probability of \( \text{Query}(\mathcal{CT}, TK_\theta, PK) \neq \bot \) is negligible by limiting \( |M| \) to less than \( |\mathbb{G}_T|^{1/4} \).

5.4.3 Security

**Theorem 5.4.1.** The above HVE construction is selectively secure under the decisional BDH and P3DH assumptions.

The proof of this theorem is easily obtained from the following five Lemma 5.4.2, 5.4.3, 5.4.4, 5.4.5, and 5.4.6. Before presenting the five lemmas, we first introduce the following four assumptions. The HVE scheme of Lee and Lee constructed in bilinear groups of composite \( n = p_1 p_2 p_3 \) order, and its security was proven under the decisional BDH, Bilinear Subgroup Decision (BSD), and the decisional C3DH assumptions [36]. In composite order bilinear groups, the decisional C3DH assumption imply the decisional C2DH assumption that was introduced in [26]. However, in prime order bilinear groups, this implication is not valid since the basis vectors for ciphertexts and tokens are different. Thus the decisional C3DH assumption for ciphertexts and the decisional C2DH assumption for tokens should be treated as differently. These assumptions in composite order bilinear groups are converted to the following Assumptions 5.4-1, 5.4-2, 5.4-3, and 5.4-4 using our conversion method.

**Assumption 5.4-1** Let \( (\mathcal{G}, \mathbb{G}_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_2}, g^{b_1}) \) be the bilinear product group of basis vectors \( b_{1,1} = (1, a_1), b_{1,2} = (1, a_2, 0), b_2 = (a_2, -1, a_1 a_2 - a_3), \) and \( b_3 = (a_1, a_3, -1) \). The Assumption 5.4-1 is stated as follows: given a challenge tuple

\[
D = \left(\mathcal{G}, \mathbb{G}_T, e\right), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_2}, g^{b_1}, (g^{b_{1,1}})^{c_1}, (g^{b_{1,2}})^{c_2}, (g^{b_{2,2}})^{c_3}, (g^{b_{1,2}})^{c_4}\) and \( T \),
\]

decides whether \( T = T_0 = e(g, g)^{c_1 c_2 c_3} \) or \( T = T_1 = e(g, g)^d \) with random choices of \( c_1, c_2, c_3, d \in \mathbb{Z}_p \).

**Assumption 5.4-2** Let \( (\mathcal{G}, \mathbb{G}_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_2}, g^{b_1}) \) be the bilinear product group of basis vectors \( b_{1,1} = (1, 0, a_1), b_{1,2} = (1, a_2, 0), b_2 = (a_2, -1, a_1 a_2 - a_3), \) and \( b_3 = (a_1, a_3, -1) \). The Assumption 5.4-2 is stated as follows: given a challenge tuple

\[
D = \left(\mathcal{G}, \mathbb{G}_T, e\right), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_2}, g^{b_1}\) and \( T \),
\]

decides whether \( T = T_0 = e((g^{b_{1,1}})^{c_1} (g^{b_{1,2}})^{c_2}, (g^{b_{2,2}})^{c_3}) \) or \( T = T_1 = e((g^{b_{1,1}})^{c_1}, (g^{b_{1,2}})^{c_2}) \) with random choices of \( c_1, c_2, c_3, c_4 \in \mathbb{Z}_p \).
Assumption 5.4-3 Let \((p, G, \mathbb{G}_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_2}, g^{b_3}\) be the bilinear product group of basis vectors \(b_{1,1} = (1, 0, a_1), b_{1,2} = (1, a_2, 0), b_2 = (a_2, -1, a_1 a_2 - a_3),\) and \(b_3 = (a_1, a_3, -1)\). The Assumption 5.4-3 is stated as follows: given a challenge tuple
\[
D = \left( (p, G, \mathbb{G}_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_2}, g^{b_3}, (g^{b_{1,1}})^{c_1}, (g^{b_{1,2}})^{c_2}, (g^{b_2})^{c_1 c_2}, (g^{b_3})^{c_1 c_2 c_3} \right) \text{ and } T,
\]
decides whether \(T = T_0 = (g^{b_{1,1}})^{c_3} (g^{b_2})^{c_1} \) or \(T = T_1 = (g^{b_{1,2}})^d (g^{b_3})^{c_2} \) with random choices of \(c_1, c_2, c_3, d \in \mathbb{Z}_p\) and \(z_1, z_2, z_3 \in \mathbb{Z}_p\).

Assumption 5.4-4 Let \((p, G, \mathbb{G}_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_2}, g^{b_3}\) be the bilinear product group of basis vectors \(b_{1,1} = (1, 0, a_1), b_{1,2} = (1, a_2, 0), b_2 = (a_2, -1, a_1 a_2 - a_3),\) and \(b_3 = (a_1, a_3, -1)\). The Assumption 5.4-4 is stated as follows: given a challenge tuple
\[
D = \left( (p, G, \mathbb{G}_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_2}, g^{b_3}, (g^{b_{1,1}})^{c_1}, (g^{b_{1,2}})^{c_2}, (g^{b_2})^{c_1} \right) \text{ and } T,
\]
decides whether \(T = T_0 = (g^{b_{1,2}})^{c_1 c_2} (g^{b_3})^{c_1} \) or \(T = T_1 = (g^{b_{1,1}})^d (g^{b_3})^{c_2} \) with random choices of \(c_1, c_2, d \in \mathbb{Z}_p\) and \(z_1, z_2, z_3 \in \mathbb{Z}_p\).

Lemma 5.4.2. The above HVE construction is selectively secure under the Assumptions 5.4-1, 5.4-2, 5.4-3, and 5.4-4.

Proof. The proof of this lemma is directly obtained from [26] since the Assumptions 5.4-1, 5.4-2, 5.4-3, and 5.4-4 in prime order bilinear groups are corresponds to the Bilinear Diffie-Hellman (BDH), Bilinear Subgroup Decision (BSD), Composite 3-party Diffie-Hellman (C3DH), and Composite 2-party Diffie-Hellman (C2DH) assumptions in composite order bilinear groups.

Lemma 5.4.3. If the decisional BDH assumption holds, then the Assumption 5.4-1 also holds.

Proof. Suppose there exists an adversary \(A\) that breaks the Assumption 5.4-1 with a non-negligible advantage. An algorithm \(B\) that solves the decisional BDH assumption using \(A\) is given: a challenge tuple \(D = ((p, G, \mathbb{G}_T, e), g, g^{c_1}, g^{c_2}, g^{c_3}) \) and \(T\) where \(T = T_0 = e(g, g)^{c_1 c_2 c_3} \) or \(T = T_1 = e(g, g)^d\). \(B\) first chooses random values \(a_1, a_2, a_3 \in \mathbb{Z}_p\) and sets
\[
\begin{align*}
g^{b_{1,1}} &= (g, g^{a_1}), \\
g^{b_{1,2}} &= (g, g^{a_2}, 1), \\
g^{b_2} &= (g^{a_2}, g^{-1}, g^{a_1 a_2 - a_3}), \\
g^{b_3} &= (g^{a_1}, g^{a_3}, g^{-1}), \\
(g^{b_{1,1}})^{c_1} &= (g^{c_1}, 1, (g^{c_1})^{a_1}), \\
(g^{b_{1,2}})^{c_2} &= (g^{c_2}, 1, (g^{c_2})^{a_2}), \\
(g^{b_2})^{c_1 c_2} &= (g^{c_3}, 1, (g^{c_3})^{a_3}).
\end{align*}
\]
Next, it gives the tuple \(D' = ((p, G, \mathbb{G}_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_2}, g^{b_3}, (g^{b_{1,1}})^{c_1}, (g^{b_{1,2}})^{c_2}, (g^{b_2})^{c_1 c_2}, (g^{b_3})^{c_1 c_2 c_3}) \) and \(T\) to \(A\). Then \(A\) outputs a guess \(\gamma\). \(B\) also outputs \(\gamma\). If the advantage of \(A\) is \(\varepsilon\), then the advantage of \(B\) is greater than \(\varepsilon\) since the distribution of the challenge tuple to \(A\) is equal to the Assumption 5.4-1.

Lemma 5.4.4. The Assumption 5.4-2 holds for all adversaries.

Proof. The equation \(e((g^{b_{1,1}})^{c_1}, (g^{b_{1,2}})^{c_2}, (g^{b_2})^{c_3}) = e((g^{b_{1,1}})^{c_1}, (g^{b_{1,2}})^{c_2})\) holds by the orthogonality of basis vectors such that \(e(g^{b_{1,1}}, g^{b_{1,2}}) = 1, e(g^{b_2}, g^{b_{1,2}}) = 1, \) and \(e(g^{b_2}, g^{b_3}) = 1.\) Therefore, any adversary can not break the Assumption 5.4-2.

Lemma 5.4.5. If the decisional P3DH assumption holds, then the Assumption 5.4-3 also holds.
Proof. Suppose there exists an adversary \( \mathcal{A} \) that breaks the Assumption 5.4-3 with a non-negligible advantage. An algorithm \( \mathcal{B} \) that solves the decisional P3DH assumption using \( \mathcal{A} \) is given: a challenge tuple \( D = ((p, \mathbb{G}, \mathbb{G}_T, e), (g, f), (g^{c_1}, f^{c_1}), (g^{c_2}, f^{c_2}), (g^{c_1 c_2}, g^{c_1}), (g^{c_1 c_2} f^{c_2}, g^{c_2})) \) and \( T = T_7 = (T_{7,1}, T_{7,2}) \) where \( T = T_0 = (g^{c_1} f^{c_2}, g^{c_3}) \) or \( T = T_1 = (g^d f^{c_2}, g^{c_3}) \). \( \mathcal{B} \) first chooses random values \( a_1, a_3 \in \mathbb{Z}_p \) and sets
\[
\begin{align*}
g^{b_1,1} &= (g, 1, g^{a_1}), \ g^{b_1,2} = (g, f, 1), \ g^{b_2} = (f, g^{-1}, f^{a_1} g^{-a_1}), \ g^{b_3} = (g^{a_1}, g^{a_3}, g^{-1}), \\
(g^{b_1,2})^{c_1} &= (g^{c_1}, f^{c_1}, 1), \ (g^{b_1,2})^{c_2} = (g^{c_2}, f^{c_2}, 1), \\
(g^{b_1,2})^{c_1 c_2} (g^{b_2})^{z_1} &= (g^{c_1 c_2}, f^{c_1}, (g^{z_1})^{-1}, (g^{c_1 c_2} f^{c_1})^{a_1} (g^{z_1})^{-a_1}), \\
(g^{b_1,2})^{c_1 c_2} (g^{b_2})^{z_2} &= (g^{c_1 c_2} f^{c_2}, (g^{z_2})^{-1}, (g^{c_1 c_2} f^{c_2})^{a_1} (g^{z_2})^{-a_1}), \\
T' &= (T_{7,1}, T_{7,2}, (T_{7,1})^{a_1} (T_{7,2})^{a_1}).
\end{align*}
\]

Intuitively, it sets \( a_2 = \text{dlog}(f) \). Next, it gives the tuple \( D' = ((p, \mathbb{G}, \mathbb{G}_T, e), g^{b_1,1}, g^{b_1,2}, g^{b_2}, g^{b_3}, (g^{b_1,2})^{c_2}, (g^{b_1,1})^{c_1 c_2} (g^{b_2})^{z_1}, (g^{b_1,1})^{c_1 c_2} (g^{b_2})^{z_2}) \) and \( T' \) to \( \mathcal{A} \). Then \( \mathcal{A} \) outputs a guess \( \gamma' \). \( \mathcal{B} \) also outputs \( \gamma' \). If the advantage of \( \mathcal{A} \) is \( \varepsilon \), then the advantage of \( \mathcal{B} \) is greater than \( \varepsilon \) since the distribution of the challenge tuple to \( \mathcal{A} \) is equal to the Assumption 5.4-3.

\begin{lemma}
If the decisional P3DH assumption holds, then the Assumption 5.4-4 also holds.
\end{lemma}

Proof. Suppose there exists an adversary \( \mathcal{A} \) that breaks the Assumption 5.4-4 with a non-negligible advantage. An algorithm \( \mathcal{B} \) that solves the decisional P3DH assumption using \( \mathcal{A} \) is given: a challenge tuple \( D = ((p, \mathbb{G}, \mathbb{G}_T, e), (g, f), (g^{c_1}, f^{c_1}), (g^{c_2}, f^{c_2}), (g^{c_1 c_2} f^{c_1}, g^{c_3}), (g^{c_1 c_2} f^{c_2}, g^{c_3})) \) and \( T = T_7 = (T_{7,1}, T_{7,2}) \) where \( T = T_0 = (g^{c_1 c_2} f^{c_2}, g^{c_3}) \) or \( T = T_1 = (g^d f^{c_2}, g^{c_3}) \). \( \mathcal{B} \) first chooses random values \( a_1, a_3 \in \mathbb{Z}_p \) and sets
\[
\begin{align*}
g^{b_1,1} &= (g, 1, f), \ g^{b_1,2} = (g, g^{a_2}, 1), \ g^{b_2} = (g^{a_2}, g^{-1}, g^{a_3}), \ g^{b_3} = (f, f^{a_2} g^{-a_1}, g^{-1}), \\
(g^{b_1,2})^{c_1} &= (g^{c_1}, f^{c_1}, 1), \ (g^{b_1,2})^{c_2} (g^{b_2})^{z_1} &= (g^{c_1 c_2}, f^{c_1}, (g^{z_1})^{-1}, (g^{c_1 c_2} f^{c_1})^{a_1} (g^{z_1})^{-a_1}), \\
(g^{b_1,2})^{c_2} (g^{b_2})^{z_2} &= (g^{c_3} f^{c_2}, (g^{z_2})^{-1}, (g^{c_3} f^{c_2})^{a_1} (g^{z_2})^{-a_1}), \\
T' &= (T_{7,1}, (T_{7,1})^{a_1} (T_{7,2})^{a_1}).
\end{align*}
\]

Intuitively, it sets \( a'_1 = \text{dlog}(f), a'_2 = a_2, a'_3 = a_1 a_2 - a_3 \) and \( c'_1 = c_1 c_2, c'_2 = c_3 \) where \( a'_1, a'_2, a'_3 \) are elements of basis vectors for the Assumption 5.4-4. Next, it gives the tuple \( D' = ((p, \mathbb{G}, \mathbb{G}_T, e), g^{b_1,1}, g^{b_1,2}, g^{b_2}, g^{b_3}, (g^{b_1,2})^{c_2}, (g^{b_1,1})^{c_1 c_2} (g^{b_2})^{z_1}, (g^{b_1,1})^{c_1 c_2} (g^{b_2})^{z_2}) \) and \( T' \) to \( \mathcal{A} \). Then \( \mathcal{A} \) outputs a guess \( \gamma' \). \( \mathcal{B} \) also outputs \( \gamma' \). If the advantage of \( \mathcal{A} \) is \( \varepsilon \), then the advantage of \( \mathcal{B} \) is greater than \( \varepsilon \) since the distribution of the challenge tuple to \( \mathcal{A} \) is equal to the Assumption 5.4-4.

\end{proof}

### 5.5 Conversion 3: SW-dHVE

In this section, we convert the delegatable HVE scheme of Shi and Waters [36] to prime order bilinear groups and prove its selective model security under the decisional BDH and P3DH assumptions.

#### 5.5.1 Construction

Let \( \Sigma \) be a finite set of attributes and let \( ?, * \) be two special symbol not in \( \Sigma \). Define \( \Sigma_{2,*} = \Sigma \cup \{?, *\} \). The symbol \( ? \) denotes a delegatable field, i.e., a field where one is allowed to fill in an arbitrary value and perform delegation. The symbol \( * \) denotes a wild-card field or “don’t care” field.
Setup(1^λ,l): The setup algorithm first generates the bilinear group \(G\) of prime order \(p\) of bit size \(\Theta(\lambda)\). It chooses random values \(a_1, a_2, a_3 \in \mathbb{Z}_p\) and sets basis vectors for bilinear product groups as \(b_{1,1} = (1, 0, a_1)\), \(b_{1,2} = (1, a_2, 0)\), \(b_2 = (a_2, -1, a_1 a_2 - a_3)\), \(b_3 = (a_1, a_3, -1)\). Next, it chooses random exponents \(v_1, w_1, w_2, \{u_i, h_i\}_{i=1}^l\), \(\alpha \in \mathbb{Z}_p\), and it computes the following values using the basis vectors:

\[
\begin{align*}
B_{1,1} &= g^{b_{1,1}}, \quad B_{1,2} = g^{b_{1,2}}, \quad B_2 = g^{b_2}, \quad B_3 = g^{b_3}, \\
g^{v_1} &= B_{1,1}^{v_1}, \quad g^{w_1} = B_{1,1}^{w_1}, \quad g^{w_2} = B_{1,2}^{w_2}, \quad \{g^{u_i} = B_{1,1}^{u_i}, \quad g^{h_i} = B_{1,1}^{h_i}\}_{i=1}^l,
\end{align*}
\]

It keeps \(g^{v_1}, \{g^{u_1}, g^{h_1}\}, g^{w_1}\); \(\{g^{w_2} i\}_{i=1}^l\) as a secret key \(SK\). Then it publishes a public key \(PK\) using random blinding values \(z_1, z_2, \ldots, z_l, \{z_{a_1}, z_{a_2}\}_{i=1}^l \in \mathbb{Z}_p\) as follows:

\[
PK = \left( B_{1,1}, B_{1,2}, B_2, B_3, V = g^{v_1} B_2^2, \quad W_1 = g^{w_1} B_2^2, \quad W_2 = g^{w_2} B_2^2, \quad \{U_i = g^{u_i} B_2^2, \quad H_i = g^{h_i} B_2^2\}_{i=1}^l, \quad \Omega = e(g^{v_1}, g^{h_1})^\alpha \right).
\]

GenToken(\(\sigma, SK, PK\)): The token generation algorithm takes as input an attribute vector \(\sigma = (\sigma_1, \ldots, \sigma_l) \in \Sigma_{\sigma, l}\) and the secret key \(SK\). Let \(S\) be the set of indexes that are not delegatable fields and wild-card fields in the vector \(\sigma\). It first selects random exponents \(r_1, r_2, \{r_{3, i}\}_{i \in S} \in \mathbb{Z}_p\) and random blinding values \(y_1, y_2, y_3, \{y_{4, i}\}_{i \in S} \in \mathbb{Z}_p\). Then it computes decryption components as:

\[
\begin{align*}
K_1 &= (g^{b_{1,2}})^\sigma (g^{w_1})^{r_1} (g^{w_2})^{r_2} \prod_{i \in S} ((g^{u_i})^\sigma g^{b_{1,1} h_i} y_i) B_3^{u_i}, \\
K_2 &= (g^{v_2})^{r_1} B_3^{u_1}, \quad K_3 = (g^{v_2})^{-2} B_3^{v_1}, \quad \{K_{4, i} = (g^{v_2})^{-r_{3, i}} B_3^{y_i}\}_{i \in S}.
\end{align*}
\]

Let \(S_{\sigma}\) be the set of indexes that are delegatable fields. It selects random exponents \(\{s_{1, j}, s_{2, j}\}, \{s_{3, j}\}_{i \in S_{\sigma}} \in \mathbb{Z}_p\) and random blinding values \(\{y_{1, j, u}, y_{1, j, h}, y_{2, j}, y_{3, j}, \{y_{4, j}\}_{i \in S_{\sigma}}\} \in \mathbb{Z}_p\). Next, it computes delegation components as:

\[
\forall j \in S_{\sigma}, \quad \begin{align*}
L_{1, j, u} &= (g^{u_{s_1, j}}) y_{3, j} B_3^{y_{4, j}}, \\
L_{1, j, h} &= (g^{w_2})^{s_{1, j}} (g^{w_2})^{s_{2, j}} \prod_{\sigma_i \in S_{\sigma}} ((g^{u_i})^\sigma g^{b_{1,1} h_i} y_i) B_3^{u_i}, \\
L_{2, j} &= (g^{v_2})^{-s_{1, j}} B_3^{v_1}, \quad L_{3, j} = (g^{v_2})^{-s_{2, j}} B_3^{v_1}, \quad \{L_{4, j, i} = (g^{v_2})^{-s_{3, j}} B_3^{y_i}\}_{i \in S_{\sigma}}.
\end{align*}
\]

Finally, it outputs a token as:

\[
\text{TK}_\sigma = \left( \begin{array}{c}
K_1, \quad K_2, \quad K_3, \quad \{K_{4, j}\}_{i \in S_{\sigma}}, \quad \{L_{1, j, u}, \quad L_{1, j, h}, \quad L_{2, j}, \quad L_{3, j}, \quad \{L_{4, j, i}\}_{i \in S_{\sigma}}\}_{j \in S_{\sigma}}
\end{array} \right).
\]

Delegate(\(\sigma', \text{TK}_\sigma, PK\)): The delegation algorithm takes as input an attribute vector \(\sigma' = (\sigma_1, \ldots, \sigma_l) \in \Sigma_{\sigma, l}\) and a token \(\text{TK}_\sigma\). Without loss of generality, we assume that \(\sigma'\) fixes only one delegatable field of \(\sigma\). It is clear that we can perform delegation on multiple fields if we have an algorithm to perform delegation on one field. Suppose \(\sigma'\) fixes the \(k\)-th index of \(\sigma\). If the \(k\)-th index of \(\sigma'\) is set to 0, that is, a wild-card field, then it can perform delegation by simply removing the delegation components that correspond to \(k\)-th index. Otherwise, that is, if the \(k\)-th index of \(\sigma'\) is set to some value in \(\Sigma\), then it perform delegation as follows.
Let $S$ be the set of indexes that are not delegatable fields and wild-card fields in the vector $\sigma'$. Note that $k \in S$. It selects random exponents $\mu, y_1, y_2, y_3, \{y_{4,i}\}_{i \in S} \in \mathbb{Z}_p$ and updates the token as

\[
\mathbf{K}_1 = K_1(L_{\sigma,1,k,h}^{\mu} B_3^{y_1}), \quad \mathbf{K}_2 = K_2 L_{2,k}^{\mu} B_3^{y_2}, \quad \mathbf{K}_3 = K_3 L_{3,k}^{\mu} B_3^{y_3},
\]

\[
\mathbf{K}_{4,k} = L_{4,k}^{\mu} B_3^{y_{4,k}}, \quad \mathbf{K}_{4,i} = K_{4,i} L_{4,k}^{\mu} B_3^{y_{4,i}} \right\}_{i \in S \setminus \{k\}}.
\]

Let $S_7$ be the set of indexes that are delegatable fields in the vector $\sigma'$. It selects random exponents $\{\tau_j, y_{1,j,us}, y_{1,j,h}, y_{2,j,us}, y_{2,j,h}, y_{3,j,us}, \{y_{4,j,i}\}_{i \in S \setminus \{j\}}\}_{j \in S_7} \in \mathbb{Z}_p$ and re-randomizes the delegation components of the token as

\[
\forall j \in S_7 : \mathbf{L}_{1,j,a} = L_{1,j,a}^{\mu} B_3^{y_1,ia}, \quad \mathbf{L}_{1,j,h} = L_{1,j,h}^{\mu} (L_{1,k,c}^{\sigma,\mu} L_{1,k,h})^{\tau_j} B_3^{y_1,jh},
\]

\[
\mathbf{L}_{2,j} = L_{2,j}^{\mu} L_{2,j}^{\tau_j} B_3^{y_2,j}, \quad \mathbf{L}_{3,j} = L_{3,j}^{\mu} L_{3,j}^{\tau_j} B_3^{y_3,j},
\]

\[
\mathbf{L}_{4,j,i} = L_{4,j,i}^{\mu} L_{4,j,i}^{\tau_j} B_3^{y_{4,j,i}}, \quad \mathbf{L}_{4,j,k} = L_{4,j,k}^{\mu} L_{4,j,k}^{\tau_j} B_3^{y_{4,j,k}} \right\}_{i \in S \setminus \{k\}}.
\]

Finally, it outputs a token as

\[
\text{TK}_{\sigma'} = \left( \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \{\mathbf{K}_{4,i}\}_{i \in S}, \{\mathbf{L}_{1,j,a}, \mathbf{L}_{1,j,h}, \mathbf{L}_{2,j}, \mathbf{L}_{3,j}, \mathbf{L}_{4,j,i}, \{\mathbf{L}_{4,j,k}\}_{i \in S \setminus \{j\}}\}_{j \in S_7} \right).
\]

Encrypt($x, M, PK$): The encryption algorithm takes as input an attribute vector $x = (x_1, \ldots, x_l) \in \mathcal{X}$, a message $M \in \mathcal{M} \subseteq \mathbb{G}_{T}$, and the public key $PK$. It first chooses a random exponent $t \in \mathbb{Z}_p$ and random blinding values $z_1, z_2, z_3, \{z_{4,i}\}_{i = 1}^l \in \mathbb{Z}_p$. Then it outputs a ciphertext as

\[
\text{CT} = \left( C_0 = \Omega^t M, \quad C_1 = V^{t} B_{1}^{z_1}, \quad C_2 = W_{1}^{\sigma} B_{2}^{z_2}, \quad C_3 = W_{2}^{\sigma} B_{3}^{z_3}, \quad \{C_{4,j} = (U_{j}^{\mu} H_{j})^{t} B_{4}^{z_{4,j}}\}_{j = 1}^{l} \right).
\]

Query($\text{CT}, \text{TK}_{\sigma}, PK$): The query algorithm takes as input a ciphertext $\text{CT}$ and a token $\text{TK}_{\sigma}$ of a vector $\sigma$. It first computes

\[
M \leftarrow C_0 \cdot \left( e(C_1, K_1) \cdot e(C_2, K_2) \cdot e(C_3, K_3) \cdot \prod_{i \in S} e(C_{4,i}, K_{4,i}) \right)^{-1}.
\]

If $M \notin \mathcal{M}$, it outputs $\perp$ indicating that the predicate $f_{\sigma}$ is not satisfied. Otherwise, it outputs $M$ indicating that the predicate $f_{\sigma}$ is satisfied.

### 5.5.2 Correctness

If $f_{\sigma}(\tilde{x}) = 1$, then the following calculation shows that $\text{Query}(\text{CT}, \text{TK}_{\sigma}, PK) = M$ by the orthogonality of basis vectors such that $e(g^{b_{1,i}} g^{b_{1,j}}) = 1, e(g^{b_{1,j}} g^{b_{2}}) = 1$, and $e(g^{b_{2}} g^{b_{3}}) = 1$.

\[
e(C_1, K_1) \cdot e(C_2, K_2) \cdot e(C_3, K_3) \cdot \prod_{i \in S} e(C_{4,i}, K_{4,i})
\]

\[
= e(g^{y_1} t, g^{y_{2,j}})^{\tau_j} \cdot \prod_{i \in S} (g^{u_{i,j}})^{\sigma,\mu} g^{b_{1i}})^{y_{3,i}}.
\]

\[
e(g^{y_{1,1}} t, g^{y_{2}})^{-t_1} \cdot e(g^{y_{2,1}} t, g^{y_{2}})^{-r_2} \cdot \prod_{i \in S} e(g^{u_{i,j}} t, g^{b_{1i}})^{t_{r_3,i}}
\]

\[
= e(g^{y_1} t, g^{b_{1,j}})^{\alpha} \cdot \prod_{i \in S} g^{y_1 t, (u_{i,j})}^{(\sigma - \chi)} t_{r_3,i} = e(g^{y_1} t, g^{b_{1,1}})^{\alpha}.
\]

Otherwise, that is $f_{\sigma}(\tilde{x}) = 0$, the probability of $\text{Query}(\text{CT}, \text{TK}_{\sigma}, PK) \neq \perp$ is negligible by limiting $|\mathcal{M}|$ to less than $|\mathbb{G}_T|^{1/4}$.
5.5.3 Security

**Theorem 5.5.1.** The above dHVE construction is selectively secure under the decisional BDH and P3DH assumptions.

The proof of this theorem is easily obtained from the following five Lemma 5.5.2, 5.5.3, 5.5.4, 5.5.5 and 5.5.6. Before presenting the five lemmas, we first introduce the following four assumptions. The HVE scheme of Shi and Waters constructed in bilinear groups of composite \( n = p_1p_2p_3 \) order, and its security was proven under the decisional BDH, Bilinear Subgroup Decision (BSD), and the decisional C3DH assumptions [36]. In composite order bilinear groups, the decisional C3DH assumption imply the decisional l-C3DH assumption that was introduced in [36]. However, in prime order bilinear groups, this implication is not valid since the basis vectors for ciphertexts and tokens are different. Thus the decisional C3DH assumption for ciphertexts and the decisional C3DH assumption for tokens should be treated as differently. These assumptions in composite order bilinear groups are converted to the following Assumptions 5.5-1, 5.5-2, 5.5-3, and 5.5-4 using our conversion method.

**Assumption 5.5-1** Let \( ((p, \mathbb{G}, \mathbb{G}_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_2}, g^{b_3}) \) be the bilinear product group of basis vectors \( b_{1,1} = (1, 0, a_1), b_{1,2} = (1, a_2, 0), b_2 = (a_2, -1, a_1a_2 - a_3), \) and \( b_3 = (a_1, a_3, -1) \). The Assumption 5.5-1 is stated as follows: given a challenge tuple

\[
D = \left( (p, \mathbb{G}, \mathbb{G}_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_2}, g^{b_3}, (g^{b_{1,1}})^{c_1}, (g^{b_{1,2}})^{c_2}, (g^{b_2})^{c_3}, (g^{b_3})^{c_4} \right) \quad \text{and} \quad T,
\]

decides whether \( T = T_0 = e(g, g)^{c_1c_2c_3} \) or \( T = T_1 = e(g, g)^d \) with random choices of \( c_1, c_2, c_3, d \in \mathbb{Z}_p \).

**Assumption 5.5-2** Let \( ((p, \mathbb{G}, \mathbb{G}_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_2}, g^{b_3}) \) be the bilinear product group of basis vectors \( b_{1,1} = (1, 0, a_1), b_{1,2} = (1, a_2, 0), b_2 = (a_2, -1, a_1a_2 - a_3), \) and \( b_3 = (a_1, a_3, -1) \). The Assumption 5.5-2 is stated as follows: given a challenge tuple

\[
D = \left( (p, \mathbb{G}, \mathbb{G}_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_2}, g^{b_3} \right) \quad \text{and} \quad T,
\]

decides whether \( T = T_0 = e((g^{b_{1,1}})^{c_1}(g^{b_{1,2}})^{c_2}(g^{b_2})^{c_3}) \) or \( T = T_1 = e((g^{b_{1,1}})^{c_1}, (g^{b_{1,2}})^{c_2}) \) with random choices of \( c_1, c_2, c_3, c_4 \in \mathbb{Z}_p \).

**Assumption 5.5-3** Let \( ((p, \mathbb{G}, \mathbb{G}_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_2}, g^{b_3}) \) be the bilinear product group of basis vectors \( b_{1,1} = (1, 0, a_1), b_{1,2} = (1, a_2, 0), b_2 = (a_2, -1, a_1a_2 - a_3), \) and \( b_3 = (a_1, a_3, -1) \). The Assumption 5.5-3 is stated as follows: given a challenge tuple

\[
D = \left( (p, \mathbb{G}, \mathbb{G}_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_2}, g^{b_3}, (g^{b_{1,1}})^{c_1}, (g^{b_{1,2}})^{c_2}, (g^{b_2})^{c_3}, (g^{b_3})^{c_4} \right) \quad \text{and} \quad T,
\]

decides whether \( T = T_0 = (g^{b_{1,1}})^{c_1}(g^{b_2})^{c_2} \) or \( T = T_1 = (g^{b_{1,1}})^d(g^{b_2})^{c_2} \) with random choices of \( c_1, c_2, c_3, d \in \mathbb{Z}_p \), and \( z_1, z_2, z_3 \in \mathbb{Z}_p \).

**Assumption 5.5-4** Let \( ((p, \mathbb{G}, \mathbb{G}_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_2}, g^{b_3}) \) be the bilinear product group of basis vectors \( b_{1,1} = (1, 0, a_1), b_{1,2} = (1, a_2, 0), b_2 = (a_2, -1, a_1a_2 - a_3), \) and \( b_3 = (a_1, a_3, -1) \). The Assumption 5.5-4 is stated as follows: given a challenge tuple

\[
D = \left( (p, \mathbb{G}, \mathbb{G}_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_2}, g^{b_3}, (g^{b_{1,1}})^{c_1}, (g^{b_{1,2}})^{c_2}, (g^{b_2})^{c_3}, (g^{b_3})^{c_4} \right) \quad \text{and} \quad T,
\]

decides whether \( T = T_0 = (g^{b_{1,2}})^{c_1}(g^{b_3})^{c_2} \) or \( T = T_1 = (g^{b_{1,2}})^d(g^{b_3})^{c_2} \) with random choices of \( c_1, c_2, c_3, d \in \mathbb{Z}_p \), and \( z_1, z_2, z_3 \in \mathbb{Z}_p \).

45
**Lemma 5.5.2.** The above dHVE construction is selectively secure under the Assumptions 5.5-1, 5.5-2, 5.5-3, and 5.5-4.

**Proof.** The proof of this lemma is directly obtained from [36] since the Assumptions 5.5-1, 5.5-2, 5.5-3, and 5.5-4 in prime order bilinear groups are correspond to the Bilinear Diffie-Hellman (BDH), Bilinear Subgroup Decision (BSD), Composite 3-party Diffie-Hellman (C3DH), and Composite 3-party Diffie-Hellman (C3DH) assumptions in composite order bilinear groups.

**Lemma 5.5.3.** If the decisional BDH assumption holds, then the Assumption 5.5-1 also holds.

**Lemma 5.5.4.** The Assumption 5.5-2 holds for all adversaries.

**Lemma 5.5.5.** If the decisional P3DH assumption holds, then the Assumption 5.5-3 also holds.

The Assumptions 5.5-1, 5.5-2, and 5.5-3 are the same as the Assumptions 5.4-1, 5.4-2, and 5.4-3. Thus we omit the proof of Lemma 5.5.3, 5.5.4, 5.5.5.

**Lemma 5.5.6.** If the decisional P3DH assumption holds, then the Assumption 5.5-4 also holds.

**Proof.** Suppose there exists an adversary \( \mathcal{A} \) that breaks the Assumption 5.5-4 with a non-negligible advantage. An algorithm \( \mathcal{B} \) that solves the decisional P3DH assumption using \( \mathcal{A} \) is given: a challenge tuple \( D = ((\hat{p}, G, G_T, e), (g, f), (g^{c_1}, f^{c_1}), (g^{c_2}, f^{c_2}), (g^{(c_1^2 c_2)}, g^{\gamma_1}), (g^{c_1 c_2 f_2}, f^{c_2}, g^{\gamma_2})) \) and \( T = T' = (T_{\gamma_1}, T_{\gamma_2}) \) where \( T = T_0 = (g^{c_1 f^{c_2}}, g^{\gamma_2}) \) or \( T = T_1 = (g^{d c_1 f^{c_2}}, g^{\gamma_2}) \). \( \mathcal{B} \) first chooses random values \( a_2, a_3 \in \mathbb{Z}_p \) and sets

\[
\begin{align*}
 &g^{b_{1,1}} = (g, 1, f),
g^{b_{1,2}} = (g, g^{a_2}, 1),
g^{b_2} = (g^{a_2}, g^{-1}, g^{a_3}),
g^{b_3} = (f, f^{a_2} g^{-a_1}, g^{-1}),
&(g^{b_{1,1} c_1}) = (g^{c_1}, 1, f^{c_1}),
&(g^{b_{1,1} c_2}) = (g^{c_2}, 1, f^{c_2}),
&(g^{b_{1,2} c_1} (g^{b_{1,2}})^{c_1}) = (g^{c_1 c_2 f^{c_1}}, (g^{c_1 c_2 f^{c_1}})^{a_2} (g^{\gamma_1})^{-a_3}, (g^{\gamma_2})^{-1}),
&(g^{b_{1,2} c_1} (g^{b_{1,2}})^{c_2}) = (g^{c_1 c_2 f^{c_2}}, (g^{c_1 c_2 f^{c_2}})^{a_2} (g^{\gamma_2})^{-a_3}, (g^{\gamma_2})^{-1}),
&T' = (T_{\gamma_1}, (T_{\gamma_1})^{a_2} (T_{\gamma_2})^{-a_3}, (T_{\gamma_2})^{-1}).
\end{align*}
\]

Intuitively, it sets \( a_1' = \text{dlog}(f), a_2' = a_2, a_3' = a_1 a_2 - a_3 \) where \( a_1', a_2', a_3' \) are elements of basis vectors for the Assumption 5.5-4. Next, it gives the tuple \( D' = ((\hat{p}, G, G_T, e), g^{b_{1,1}}, g^{b_{1,2}}, g^{b_2}, g^{b_3}, (g^{b_{1,1}})^{c_1}, (g^{b_{1,1}})^{c_2}, (g^{b_{1,2}})^{c_1} (g^{b_{1,2}})^{c_2}, (g^{b_{1,2}})^{c_1} (g^{b_{1,2}})^{c_2} (g^{b_{1,2}})^{c_2}) \) and \( T' \) to \( \mathcal{A} \). Then \( \mathcal{A} \) outputs a guess \( \gamma' \). \( \mathcal{B} \) also outputs \( \gamma' \). If the advantage of \( \mathcal{A} \) is \( \varepsilon \), then the advantage of \( \mathcal{B} \) is greater than \( \varepsilon \) since the distribution of the challenge tuple to \( \mathcal{A} \) is equal to the Assumption 5.5-4. \( \square \)
Chapter 6

Fully Secure HVE with Short Tokens

6.1 Overview

In this chapter, we propose a fully secure HVE scheme with short tokens. Our construction based on composite order bilinear groups of products of four primes and proved under four static assumptions.

The full security model is the right security model for predicate encryption. However, it is not easy to provide full security model with reasonable security reduction loss. Recently, Waters proposed a novel proof technique called the dual system encryption [40]. In the dual system encryption, the security proof consists of hybrid games that change the original security game to a new game that can not be distinguishable from the adversary’s view.

The dual system encryption was very successful to prove the full security model of hierarchical identity-based encryption, attribute-based encryption, and public-key broadcast encryption. However, this technique does not work well in predicate encryption. Main reason of this difficulty is that predicate encryption should provide the attribute hiding property that guarantees the anonymity of the ciphertexts and the adversary of predicate encryption can query a predicate that satisfies with the challenge ciphertext.

To overcome this problem, we restrict the adversary’s capability as he can only query predicates \( f \) such that \( f(x_0) = f(x_1) = 0 \) where \( x_0, x_1 \) are the challenge vectors. That is, the adversary can not query a predicate that satisfies with the challenge ciphertext.

6.2 HVE in Composite Order Groups

In this section, we construct an efficient HVE scheme in composite order bilinear groups and prove its full model security under static assumptions.

6.2.1 Construction

Let \( \Sigma = \mathbb{Z}_m \) for some integer \( m \) and set \( \Sigma_* = \mathbb{Z}_m \cup \{\ast\} \). Our scheme is described as follows.

**Setup(1^\lambda, l):** The setup algorithm first generates the bilinear group \( G \) of composite order \( n = p_1 p_2 p_3 p_4 \) where \( p_1, p_2, p_3 \) and \( p_4 \) are random primes of bit size \( \Theta(\lambda) \). It chooses random elements \( g, \{u_i, h_i\}_{i=1}^l \in \mathbb{G}_{p_1}, Z \in \mathbb{G}_{p_2}, Y \in \mathbb{G}_{p_3} \) and a random exponent \( \alpha \in \mathbb{Z}_{p_1} \). It keeps \( g, g^\alpha, \{u_i, h_i\}_{i=1}^l, Y \) as a master key.
MK. Next, it selects random elements \( Z_v, \{ Z_{u,i}, Z_{h,i} \}_{i=1}^l \in \mathbb{G}_{p_2} \) and publishes a public key \( PK \) as
\[
PK = \left( V = gZ_v, \{ U_i = u_iZ_{u,i}, H_i = h_iZ_{h,i} \}_{i=1}^l, Z, \Omega = e(g,g)^\alpha \right).
\]

GenToken(\( \sigma, SK, PK \)): The token generation algorithm takes as input a vector \( \sigma = (\sigma_1, \ldots, \sigma_l) \in \Sigma^l \) and the secret key \( SK \). Let \( S \) be the set of indexes that are not wild cards in the vector \( \sigma \). It selects a random exponent \( r \in \mathbb{Z}_n \) and random elements \( Y_1, Y_2 \in \mathbb{G}_{p_2} \). Then it outputs a token as
\[
TK_\sigma = \left( K_1 = g^{\alpha} \left( \prod_{i \in S} u_i^{\sigma_i} h_i \right)^{Y_1}, K_2 = g^{\sigma_2} Y_2 \right).
\]

Encrypt(x, M, PK): The encryption algorithm takes as input a vector \( x = (x_1, \ldots, x_l) \in \Sigma^l \), a message \( M \in \mathcal{M} \subseteq \mathbb{G}_T \), and the public key \( PK \). It first chooses a random exponent \( r \in \mathbb{Z}_n \) and random elements \( Z_1, \{ Z_{2,i} \}_{i=1}^l \in \mathbb{G}_{p_2} \). Then it outputs a ciphertext as
\[
CT = \left( C_0 = \Omega^r M, C_1 = V^r Z_1, \{ C_{2,i} = (U_i^{c_{i2}} H_i)^r Z_{2,i} \}_{i=1}^l \right).
\]

Query(CT, TK_\sigma, PK): The query algorithm takes as input a ciphertext \( CT \) and a token \( TK_\sigma \) of a vector \( \sigma \). It first computes
\[
M \leftarrow C_0 \cdot e(C_1, K_1)^{-1} \cdot e(\prod_{i \in S} C_{2,i}, K_2).
\]
If \( M \notin \mathcal{M} \), it outputs \( \perp \) indicating that the predicate \( f_\sigma \) is not satisfied. Otherwise, it outputs \( M \) indicating that the predicate \( f_\sigma \) is satisfied.

### 6.2.2 Correctness

If \( f_\sigma(x) = 1 \), then the following calculations shows that \( \text{Query}(CT, TK_\sigma, PK) = M \) as
\[
e(C_1, K_1)^{-1} \cdot e(\prod_{i \in S} C_{2,i}, K_2) = e(V^r Z_1, g^{\alpha} (\prod_{i \in S} u_i^{\sigma_i} h_i)^r Y_1)^{-1} \cdot e(\prod_{i \in S} (U_i^{c_{i2}} H_i)^r Z_{2,i}, g^{\sigma_2} Y_2)
= e(g^r, g^{\alpha})^{-1} \cdot e((\prod_{i \in S} u_i^{(-\sigma_i + x_i)} h_i)^r, g^r) = e(g, g)^{-\alpha}.
\]

Otherwise, that is \( f_\sigma(x) = 0 \), then the probability of \( \text{Query}(CT, TK_\sigma, PK) \neq \perp \) is negligible by limiting \( |\mathcal{M}| \) to less than \( |\mathcal{G}_T|^{1/4} \).

### 6.2.3 Complexity Assumptions

We introduce four static assumptions under composite order bilinear groups.

**Assumption 1 (Subgroup Decision Assumption)** Let \( (n, G, \mathbb{G}_T, e) \) be a description of the bilinear group of composite order \( n = p_1 p_2 p_3 p_4 \). Let \( g_{p_1}, g_{p_2}, g_{p_3}, g_{p_4} \) be generators of subgroups of order \( p_1, p_2, p_3, p_4 \) of \( G \) respectively. The Assumption 1 is stated as follows: given a challenge tuple
\[
D = ((n, G, \mathbb{G}_T, e), g_{p_1}, g_{p_2}, g_{p_3}) \text{ and } T,
\]
decides whether \( T = T_0 = Z_1 \in \mathbb{G}_{p_2} \) or \( T = T_1 = Z_1 R_1 \in \mathbb{G}_{p_2 p_3} \) with random choices of \( Z_1 \in \mathbb{G}_{p_2} \), \( R_1 \in \mathbb{G}_{p_3} \).
**Assumption 2** Let \((n, G, G_T, e)\) be a description of the bilinear group of composite order \(n = p_1p_2p_3p_4\). Let \(g_{p_1}, g_{p_2}, g_{p_3}, g_{p_4}\) be generators of subgroups of order \(p_1, p_2, p_3, p_4\) of \(G\) respectively. The Assumption 2 is stated as follows: given a challenge tuple
\[
D = ((n, G, G_T, e), g_{p_1}, g_{p_2}, g_{p_3}, X_1R_1, Y_1R_2) \text{ and } T,
\]
decides whether \(T = T_0 = X_2Y_2\) or \(T = T_1 = X_2Y_2R_3\) with random choices of \(X_1, X_2 \in G_{p_1}, Y_1, Y_2 \in G_{p_3}, R_1, R_2, R_3 \in G_{p_4}\).

**Assumption 3** Let \((n, G, G_T, e)\) be a description of the bilinear group of composite order \(n = p_1p_2p_3p_4\). Let \(g_{p_1}, g_{p_2}, g_{p_3}, g_{p_4}\) be generators of subgroups of order \(p_1, p_2, p_3, p_4\) of \(G\) respectively. The Assumption 2 is stated as follows: given a challenge tuple
\[
D = ((n, G, G_T, e), g_{p_1}, g_{p_2}, g_{p_3}, g_{p_4}, g^a_{p_1}, Z_1, g^a_{p_1}Y_1R_1, Y_2R_1, g^b_{p_1}Z_2R_2) \text{ and } T,
\]
decides whether \(T = T_0 = g_{p_1}^{ab}Z_3R_3\) or \(T = T_1 = g_{p_1}^aZ_3R_3\) with random choices of \(a, b, c \in \mathbb{Z}_{p_1}, Z_1, Z_2, Z_3 \in G_{p_2}, Y_1, Y_2 \in G_{p_3}, R_1, R_2, R_3 \in G_{p_4}\).

**Assumption 4** Let \((n, G, G_T, e)\) be a description of the bilinear group of composite order \(n = p_1p_2p_3p_4\). Let \(g_{p_1}, g_{p_2}, g_{p_3}, g_{p_4}\) be generators of subgroups of order \(p_1, p_2, p_3, p_4\) of \(G\) respectively. The Assumption 3 is stated as follows: given a challenge tuple
\[
D = ((n, G, G_T, e), g_{p_1}, g_{p_2}, g_{p_3}, g_{p_4}, g^a_{p_1}, Z_1, g^a_{p_1}Y_1R_1, Y_2R_1, g^b_{p_1}Z_2R_2) \text{ and } T,
\]
decides whether \(T = T_0 = e(g_{p_1}, g_{p_1})^{ab}\) or \(T = T_1 = e(g_{p_1}, g_{p_1})^c\) with random choices of \(a, b, c, d \in \mathbb{Z}_{p_1}, R_1, R_2 \in G_{p_4}\).

### 6.2.4 Security

We describe a semi-functional ciphertext and a semi-functional token. They are not used in a real system, but they are used in the proof of its security. We let \(g_{p_4}\) be a generator of the subgroup \(G_{p_4}\). Let \((K'_1, K'_2)\) be a normal token and \(y, z_k\) be random exponents in \(\mathbb{Z}_n\). Then the semi-functional token is generated as
\[
\text{semi-SK}_\sigma = \left\{ K_1 = K'_1 \cdot g_{p_4}^{x_2}, K_2 = K'_2 \cdot g_{p_4}^{y} \right\}.
\]
Let \((C'_0, C'_1, \{C'_{2,i}\})\) be a normal ciphertext and \(x, z_{c,1}, \ldots, z_{c,l}\) be random exponents in \(\mathbb{Z}_n\). Then the semi-functional ciphertext is generated as
\[
\text{semi-CT} = \left( C_0 = C'_0, C_1 = C'_1 \cdot g_{p_4}^{x_1}, \{C_{2,i} = C'_{2,i} \cdot g_{p_4}^{x_{c,i}}\}_{i=1} \right).
\]
Note that if a semi-functional token is used to decrypt a semi-functional ciphertext, the decrypt algorithm will output the blinding factor multiplied by the additional term \(e(g_{p_4}, g_{p_4})^{y_0(x_2 - \sum_{i \in S} z_{c,i})}\). If \(z_k = \sum_{i \in S} z_{c,i}\), the the decrypt algorithm will still work.

**Theorem 6.2.1.** The above HVE construction is fully secure (match revealing) under the Assumptions 1, 2, 3, and 4.

**Proof.** The proof uses a sequence of games. The first game will be the original security game and the last one will be a game such that the adversary has no advantage. We define the games as follows.
Game$_0$. This game is the original full security game. Note that the private keys and the challenge ciphertext are normal.

Game$_1$. This game is almost identical to Game$_0$ except that the challenge ciphertext of a challenge vector $x_{\gamma}$ is semi-functional.

Game$_2$. This game is the same with the Game$_1$ except that the tokens will be semi-functional. At this moment, the tokens and the challenge ciphertexts are all semi-functional.

Game$_3$. In this game we will replace the challenge semi-functional ciphertext components $\{C_{2,l}\}_{l=1}^4$ to random elements in $\mathbb{G}_{p_1,p_2,p_4}$. In this case, the challenge ciphertext gives no information about the challenge vector $x_{\gamma}$.

Game$_4$. We now define a new game Game$_4$. This game differs from Game$_3$ in that the semi-functional challenge ciphertext component $C_0$ is replaced by a random element in $\mathbb{G}_{T,p_1}$. Note that in Game$_4$, the challenge ciphertext gives no information about the vector $x_{\gamma}$ and the encrypted message $M_{\gamma}$. Therefore, the adversary can win this game with probability at most $1/2$.

Through the following four lemmas, we prove that it is hard to distinguish Game$_{i-1}$ from Game$_i$ under the given assumptions. Thus, the proof is easily obtained by the following four lemmas. This completes our proof.

**Lemma 6.2.2.** If the Assumption 1 holds, then no polynomial-time adversary can distinguish between Game$_0$ and Game$_1$ with a non-negligible advantage.

**Proof.** Suppose there exists an adversary $A$ that distinguishes between Game$_0$ and Game$_1$ with a non-negligible advantage. The simulator $B$ that solves the Assumption 1 using $A$ is given: a challenge tuple $D = ((n,\mathbb{G},\mathbb{G}_T,e),g_{p_1},g_{p_2},g_{p_3})$ and $T$ where $T = Z_1 \in \mathbb{G}_{p_2}$ or $T = Z_1 R_1 \in \mathbb{G}_{p_2,p_4}$. Then $B$ that interacts with $A$ is described as follows.

**Setup:** $B$ first chooses random elements $\{u_i,h_i\}_{i=1}^l \in \mathbb{G}_{p_1}$ and a random exponent $\alpha \in \mathbb{Z}_n$. It selects random elements $Z_v,\{Z_{u,i},Z_{h,i}\}_{i=1}^l \in \mathbb{G}_{p_2}$ and publishes a public key as

$$V = g_{p_2} Z_v, \{U_i = u_i Z_{u,i}, H_i = h_i Z_{h,i}\}_{i=1}^l, Z = g_{p_2}, \Omega = e(g_{p_1},g_{p_1})^\alpha.$$  

**Query 1:** $A$ adaptively requests a token query. $B$ simply runs the token generation algorithm to create a normal token using the master key. Note that it can only create the normal tokens since it does not know $g_{p_1}$.

**Challenge:** $A$ submits two vector $x_0, x_1$ and two messages $M_0, M_1$. $B$ flips a random coin $\gamma$ internally, and it chooses random exponents $t,\{z_{c,i}\}_{i=1}^l \in \mathbb{Z}_n$. Then it outputs a ciphertext using random elements $\{Z'_{2,i}\}_{i=1}^l \in \mathbb{G}_{p_2}$ as

$$C_0 = \Omega^t M_\gamma, C_1 = V^t T, \{C_{2,i} = (U_i^{z_{c,i}} H_i)^t T^{z_{c,i}} Z'_{2,i}\}_{i=1}^l.$$  

If $T = Z_1 \in \mathbb{G}_{p_2}$, then $B$ is playing Game$_0$. Otherwise, it is playing Game$_1$. Note that it implicitly sets $g_{p_4}^{t} = R_1$.

**Query 2:** Same as Query Phase 1.

**Guess:** $A$ outputs a guess $\gamma'$. If $\gamma = \gamma'$, it outputs 0. Otherwise, it outputs 1.

This completes our proof.
Lemma 6.2.3. If the Assumption 2 holds, then no polynomial-time adversary can distinguish between $\text{Game}_1$ and $\text{Game}_2$ with a non-negligible advantage.

Proof. Suppose that an adversary makes at most $q$ private key queries. We define a sequence of games $\text{Game}_{1,0}, \text{Game}_{1,1}, \ldots, \text{Game}_{1,q}$ where $\text{Game}_{1,0} = \text{Game}_1$. In $\text{Game}_{1,i}$, for all $j$-th private key query such that $j > i$, a normal private key is given to the adversary. However, for all $j$-th private key query such that $j \leq i$, a semi-functional private key is given to the adversary. It is obvious that $\text{Game}_{1,q}$ is equal with $\text{Game}_2$.

Suppose there exists an adversary $A$ that distinguishes between $\text{Game}_{1,k-1}$ and $\text{Game}_{1,k}$ with a non-negligible advantage. A simulator $B$ that solves the Assumption 2 using $A$ is given: a challenge tuple $D = ((n, \mathbb{Z}, \mathbb{Z}, e), g_{p_1}, g_{p_2}, g_{p_3}, X_1, R_1, Y_1, R_2)$ and $T$ where $T = X_2Y_2$ or $T = X_2Y_2R_3$. Then $B$ that interacts with $A$ is described as follows.

Setup: $B$ first chooses random exponents $\{u_i, h_i\}_{i=1}^l, \alpha \in \mathbb{Z}_n$ and sets $\{u_i = g_{p_1}^{u_i}, h_i = g_{p_1}^{h_i}\}_{i=1}^l, Z = g_{p_2}, Y = g_{p_3}$. It selects random elements $Z_v, \{Z_{u,i}, Z_h,i\}_{i=1}^l \in \mathbb{Z}_{p_2}$ and publishes a public key as

$$V = g_{p_1}^{Z_v}, \{U_i = u_iZ_{u,i}, H_i = h_iZ_{h,i}\}_{i=1}^l, Z, \Omega = e(g_{p_1}, g_{p_1})^\alpha.$$

Query 1: $A$ adaptively requests a token query for a vector $\sigma = (\sigma_1, \ldots, \sigma_l) \in \Sigma^l$. If this is a $\rho$-th token query, then $B$ handles this query as follows.

Case $\rho < k$: It selects random exponents $r, y, z_k \in \mathbb{Z}_n$. Then it chooses random elements $Y_1', Y_2' \in \mathbb{Z}_{p_3}$ and outputs a semi-functional token as

$$K_1 = g_{p_1}^{\alpha} \left( \prod_{i \in S} u_i^{\sigma_i} h_i \right)^r (Y_1R_2)^{yz} Y_1', K_2 = g_{p_1}^{r} (Y_1R_2)^{Y_1'}.$$

Case $\rho = k$: It selects a random element $Y_1' \in \mathbb{Z}_{p_3}$ and outputs a token as

$$K_1 = g_{p_1}^{\alpha} T \sum_{i \in S} u_i^{\sigma_i} h_i Y_1', K_2 = T.$$

If $T = X_2Y_2$, then $B$ is playing $\text{Game}_{1,k-1}$. Otherwise, it is playing $\text{Game}_{1,k}$. Note that it implicitly sets $r = d\log(X_2), y = d\log(R_3)$, and $z_k = \sum_{i \in S}(u_i^{\sigma_i} + h_i)$. It is obvious that the distribution of token is correct as follows

$$g_{p_1}^{\alpha} \left( \prod_{i \in S} u_i^{\sigma_i} h_i \right)^r Y_1 = g_{p_1}^{\alpha} \sum_{i \in S} (u_i^{\sigma_i} + h_i) r Y_1 = g_{p_1}^{\alpha} (X_2Y_2) \sum_{i \in S} (u_i^{\sigma_i} + h_i) Y_1', g_{p_1}^{r} Y_2 = X_2Y_2.$$

Case $\rho > k$: It simply runs the token generation algorithm to create a normal token since it knows the master key.

Challenge: $A$ submits two vectors $x_0, x_1$ and two messages $M_0, M_1$. $B$ flips a random coin $\gamma$ internally and selects random elements $Z_1', Z_2', \{Z_{3,i}\}_{i=1}^l \in \mathbb{Z}_{p_2}$. Then it outputs a semi-functional ciphertext as

$$C_0 = e(X_1R_1, g_{p_1})^\alpha M_T, C_1 = (X_1R_1)Z_1', \{C_{2,i} = (X_1R_1)^{u_i\sigma_j + h_i}Z_{3,i}\}_{i=1}^l.$$

Note that it by implicitly sets $t = d\log(X_1), x = d\log(R_1)$, and $z_{c,i} = u_i^{\sigma_j} + h_i$.

Query 2: Same as Query Phase 1.
**Guess:** \( \mathcal{A} \) outputs a guess \( \gamma' \). If \( \gamma = \gamma' \), it outputs 0. Otherwise, it outputs 1.

The paradox of dual system encryption is solved since \( \{z_{c,i}\} \) of the ciphertext with a vector \( x \) and \( z_k \) of the \( k \)-th token with a vector \( \sigma \) have the relation \( z_k = \sum_{i \in S} z_{c,i} \) if \( f_\sigma(x) = 1 \). Additionally, the adversary cannot detect any relationship between \( \{z_{c,i}\} \) of the ciphertext and \( z_k \) of the \( k \)-th token since the function \( u'_i \sigma_i + h'_i \) is a pairwise independent function. This completes our proof. \( \square \)

**Lemma 6.2.4.** If the Assumption 3 holds, then no polynomial-time adversary can distinguish between \( \text{Game}_2 \) and \( \text{Game}_3 \) with a non-negligible advantage.

**Proof.** For the proof of this lemma, we define a sequence of games \( \text{Game}_{2,0}, \text{Game}_{2,1}, \ldots, \text{Game}_{2,l} \) where \( \text{Game}_{2,0} = \text{Game}_2 \). In \( \text{Game}_{2,i} \), the semi-functional ciphertext components \( \{C_{2,j}\}_{j=1}^3 \) are replaced by random elements in \( \mathbb{G}_{p_1,p_2} \). It is obvious that \( \text{Game}_{2,l} \) is equal with \( \text{Game}_3 \).

Suppose there exists an adversary \( \mathcal{A} \) that distinguishes between \( \text{Game}_{2,k-1} \) and \( \text{Game}_{2,k} \) with a non-negligible advantage. A simulator \( \mathcal{B} \) that solves the Assumption 3 using \( \mathcal{A} \) is given: a challenge tuple \( D = ((n, \mathbb{G}, \mathbb{G}_T, e), g_{p_1}, g_{p_2}, g_{p_3}, g_{p_4}, g_{p_5} \mathbb{Z}_1, g_{p_6} Y_1, Y_2, 1, g_{p_5} Z_1, g_{p_5} Z_2) \) and \( T \) where \( T = g_{p_5}^{a} Z_3 \) or \( T = g_{p_4}^{c} Z_3 \). Then \( \mathcal{B} \) that interacts with \( \mathcal{A} \) is described as follows.

**Setup:** \( \mathcal{B} \) first chooses random exponents \( \{u'_i, h'_i\}_{i=1}^l, \alpha \in \mathbb{Z}_{n} \). It selects \( Z_v, \{Z_{u,i}, Z_{h,i}\}_{i=1}^l \in \mathbb{G}_{p_2} \), and publishes a public key as

\[
V = g_{p_1} Z_v, \forall i \leq k : U_i = (g_{p_1}^a Z_1)^{u'_i} Z_{u,i}, \forall i > k : U_i = g_{p_1}^{u'_i} Z_{u,i},
\]

\[
\{H_i = g_{p_1}^{h'_i} Z_{h,i}\}_{i=1}^l, Z = g_{p_2}, \Omega = e(g_{p_1}, g_{p_1})^\alpha.
\]

**Query 1:** \( \mathcal{A} \) adaptively requests a token query for a vector \( \sigma = (\sigma_1, \ldots, \sigma_l) \). \( \mathcal{B} \) selects random exponents \( r, w \in \mathbb{Z}_n \), and random elements \( Y_1, Y_2 \in \mathbb{G}_{p_3} \). Then it outputs a semi-functional token as

\[
K_1 = g_{p_1}^{a} \left( \prod_{i \in S_1} (g_{p_1}^{a Y_1 i})^{u'_i} h'_i \right) \left( \prod_{i \in S_2} (g_{p_1}^{a Y_2 i})^{u'_i} h'_i \right)^{r w} y_1, K_2 = g_{p_1}^{a} (Y_2 Y_1)^{r w} y_2.
\]

Note that it implicitly sets \( y = dlog(R_1) r \) and \( z_k = \sum_{i \in S_1} u'_i \sigma_i + w \).

**Challenge:** \( \mathcal{A} \) submits two vectors \( x_0, x_1 \) and two messages \( M_0, M_1 \). \( \mathcal{B} \) flips a random coin \( \gamma \) internally, and it chooses a random exponent \( t \in \mathbb{Z}_n \), random elements \( Z_1, Z_2 \in \mathbb{G}_{p_2} \). Then it outputs a semi-functional ciphertext as

\[
C_0 = e(g_{p_1}^b Z_2 R_2, g_{p_1})^a M_\gamma, C_1 = (g_{p_1}^b Z_2 R_2)^{u'_i} Z_{1,i}, \forall i < k : C_{2,i} = B_1(Z_3 R_3)^{Z_{1,i}},
\]

\[
C_{2,k} = T^{u'_i \sigma_i + h'_i} Z_{2,i}, \forall i > k : C_{2,i} = (g_{p_1}^b Z_2 R_2)^{u'_i} Z_{2,i}.
\]

If \( T = g_{p_4}^{a} Z_3 \), then \( \mathcal{B} \) is playing \( \text{Game}_2 \). Otherwise, it is playing \( \text{Game}_3 \). Note that it implicitly sets \( t = b, x = dlog(R_3), \) and \( z_{c,i} = u'_i \sigma_i + h'_i \).

**Query 2:** Same as Query Phase 1.

**Guess:** \( \mathcal{A} \) outputs a guess \( \gamma' \). If \( \gamma = \gamma' \), it outputs 0. Otherwise, it outputs 1.

This completes our proof. \( \square \)
Lemma 6.2.5. If the Assumption 4 holds, then no polynomial-time adversary can distinguish between Game\textsubscript{3} and Game\textsubscript{4} with a non-negligible advantage.

Proof. Suppose there exists an adversary \( A \) that distinguishes between Game\textsubscript{3} and Game\textsubscript{4} with a non-negligible advantage. A simulator \( B \) that solves the Assumption 3 using \( A \) is given: a challenge tuple \( D = ((n, G, \mathbb{G}_T, e), g_{p_1}, g_{p_2}, g_{p_3}, g_{p_4}^a R_1, g_{p_4}^b R_2) \) and \( T \) where \( T = e(g_{p_4}^a, g_{p_4}^b) \) or \( T = e(g_{p_4}^a, g_{p_4}^b)^c \). Then \( B \) that interacts with \( A \) is described as follows.

Setup: \( B \) first chooses random elements \( \{u_i, h_i\}_{i=1}^l \in \mathbb{G}_{p_1} \). It implicitly sets \( \alpha = a \) and publishes a public key using \( Z, \{Z_{u,i}, Z_{h,i}\}_{i=1}^l \in \mathbb{G}_{p_2} \) as

\[
V = g_{p_1} Z, \{U_i = u_i Z_{u,i}, H_i = h_i Z_{h,i}\}, Z = g_{p_2}, \Omega = e(g_{p_1}, g_{p_1}^a R_1).
\]

Query 1: \( A \) adaptively requests a token query for a vector \( \sigma \). \( B \) selects random exponents \( r, z'_k \in \mathbb{Z}_n \) and a random element \( R'_1 \in \mathbb{G}_{p_2} \). Then it outputs a semi-functional token using random elements \( Y'_1, Y'_2 \in \mathbb{G}_{p_3} \) as

\[
K_1 = (g_{p_1}^a R_1) (\prod_{i \in S} u_i^{g_{p_1}^a h_i})^y (R'_1)^{z'_k} Y'_1, K_2 = g_{p_1}^b (R'_1) Y'_2.
\]

Note that it implicitly sets \( y = \text{dlog}(R'_1) \) and \( z'_k = \text{dlog}(R_1)/\text{dlog}(R'_1) + z'_k \).

Challenge: \( A \) submits two vectors \( x_0, x_1 \) and two messages \( M_0, M_1 \). \( B \) flips a random coin \( \gamma \) internally, and it chooses a random exponent \( \{w_i\}_{i=1}^l \in \mathbb{Z}_n \) and random elements \( Z'_1, \{Z_{2,i}\}_{i=1}^l \in \mathbb{G}_{p_2} \). Then it outputs a semi-functional ciphertext with randomized \( \{C_{2,i}\} \) components by implicitly setting \( t = b \) as

\[
C_0 = TM_\gamma, C_1 = (g_{p_1}^b R_2) Z'_1, \{C_{2,i} = (g_{p_1}^b R_2)^{w_i} Z'_{2,i}\}_{i=1}^l.
\]

If \( T = e(g_{p_1}^a, g_{p_1}^b) \), then \( B \) is playing Game\textsubscript{3}. Otherwise, it is playing Game\textsubscript{4}.

Query 2: Same as Query Phase 1.

Guess: \( A \) outputs a guess \( \gamma' \). If \( \gamma = \gamma' \), it outputs 0. Otherwise, it outputs 1.

This completes our proof. \( \square \)
Applications

In this chapter, we show that the HVE scheme supports conjunctive equality, conjunctive comparison, conjunctive range, and conjunctive range queries on encrypted data. The constructions of this chapter are based on [13].

7.1 Conjunctive Equality Queries

It is trivial to construct a searchable encryption system that supports conjunctive equality queries since the HVE scheme naturally supports conjunctive equality queries. Therefore, we omits the construction.

7.2 Conjunctive Comparison Queries

Let $\Sigma_{01} = \{0, 1\}$ and $\Sigma_{01^*} = \{0, 1, *\}$. Let $(\text{Setup}_{\text{HVE}}, \text{GenToken}_{\text{HVE}}, \text{Encrypt}_{\text{HVE}}, \text{Query}_{\text{HVE}})$ be a secure HVE scheme over $\Sigma_{01}^{nw}$ where $l = nw$. The searchable encryption for conjunctive comparison queries is described as follows.

$\text{Setup}(\lambda, n, w)$: The setup runs $\text{Setup}_{\text{HVE}}(1^\lambda, nw)$.

$\text{GenToken}(f_a, \text{SK}, \text{PK})$: The token generation algorithm takes as input a predicate with a vector $a = (a_1, \ldots, a_w) \in \{1, \ldots, n\}^w$ and the secret key $\text{SK}$. It first defines $\sigma_i(a) = (\sigma_i, j) \in \Sigma_{01}^{nw}$ as follows:

$$ \sigma_i, j = \begin{cases} 1 & \text{if } x_i = j, \\ * & \text{otherwise} \end{cases} $$

It outputs $\text{GenToken}_{\text{HVE}}(\sigma_i(a), \text{SK}, \text{PK})$ where the token size is $O(w)$.

$\text{Encrypt}(b, M, \text{PK})$: The encryption algorithm takes as input a vector $b = (b_1, \ldots, b_w) \in \{1, \ldots, n\}^w$, a message $M \in \mathcal{M}$, and the public key $\text{PK}$. It first defines a vector $x(b) = (x_i, j) \in \Sigma_{01}^{nw}$ as follows:

$$ x_i, j = \begin{cases} 1 & \text{if } j \geq x_i, \\ 0 & \text{otherwise} \end{cases} $$

Then it outputs $\text{Encrypt}_{\text{HVE}}(\sigma, M, \text{PK})$ where the ciphertext size is $O(nw)$.

$\text{Query}(\text{CT}, \text{TK}_a, \text{PK})$: The query algorithm outputs $\text{Query}_{\text{HVE}}(\text{CT}, \text{TK}_a)$. 

54
7.3 Conjunctive Range Queries

In previous section, we constructed a searchable encryption system that support comparison queries such that \( x \leq a \) where the ciphertext contains \( x \) and the token contains \( a \). It is easy to support comparison queries such that \( x \geq b \) by changing bit value of the ciphertext. Therefore, we can construct a searchable encryption system that support range queries by combining two comparison queries as \( x \leq a \land x \geq b \) where the ciphertext contains the pair \((x, x)\).

7.4 Subset Queries

Let \( T \) be a set of size \( n \). For a subset \( A \subseteq T \), we define a subset predicate as follows:

\[
   f_A(x) = \begin{cases} 
   1 & \text{if } x \in A, \\
   0 & \text{otherwise} 
   \end{cases}
\]

The conjunctive subset predicates are naturally defined. Let \( \Sigma_{01} = \{0, 1\} \) and \( \Sigma_{01*} = \{0, 1, *\} \). Let \((\text{Setup}_{HVE}, \text{GenToken}_{HVE}, \text{Encrypt}_{HVE}, \text{Query}_{HVE})\) be a secure HVE scheme over \( \Sigma_{01}^nw \) where \( l = nw \). The searchable encryption for conjunctive subset queries is described as follows.

\(\text{Setup}(1^\lambda, n, w)\): The setup runs \(\text{Setup}_{HVE}(1^\lambda, nw)\).

\(\text{GenToken}(f_A, SK, PK)\): The token generation algorithm takes as input a predicate with a vector \( A = (A_1, \ldots, A_w) \) and the secret key \( SK \). It first defines \( \sigma_*(A) = (\sigma_{i,j}) \in \Sigma_{01*}^nw \) as follows:

\[
   \sigma_{i,j} = \begin{cases} 
   0 & \text{if } j \notin A_i, \\
   * & \text{otherwise} 
   \end{cases}
\]

It outputs \(\text{GenToken}_{HVE}(\sigma_*(A), SK, PK)\) where the token size is \(O(nw)\).

\(\text{Encrypt}(b, M, PK)\): The encryption algorithm takes as input a vector \( b = (b_1, \ldots, b_w) \in T^w \), a message \( M \in \mathcal{M} \), and the public key \( PK \). It first defines a vector \( x(b) = (x_{i,j}) \in \Sigma_{01}^nw \) as follows:

\[
   x_{i,j} = \begin{cases} 
   1 & \text{if } x_i = j, \\
   0 & \text{otherwise} 
   \end{cases}
\]

Then it outputs \(\text{Encrypt}_{HVE}(x, M, PK)\) where the ciphertext size is \(O(nw)\).

\(\text{Query}(CT, TK_A, PK)\): The query algorithm outputs \(\text{Query}_{HVE}(CT, TK_A)\).
Chapter 8

Generic Group Model

8.1 Overview

In this chapter, we prove that the new assumption of this thesis is secure under the generic group model. The generic group model was introduced by Shoup \[37\]. The generic group model is a tool for analyzing generic algorithms that work independently of the group representation. In the generic group model, an adversary is given a random encoding of a group element or an arbitrary index of a group element instead of an actual representation of a group element. Thus, the adversary performs group operations through oracles that provided by a simulator, and the adversary only can check the equality of group elements. The detailed explanation of the generic group model is given in \[6, 24\].

The master theorems that can be used for the analysis of assumptions in bilinear groups were presented in \[6, 16, 24\]. However, the new assumption of this paper can not be analyzed by the previous master theorems. The reason of this difficulty is that the new assumption is based on symmetric bilinear groups of prime order, the target group of our assumption is \(G\) instead of \(G_T\), and the target consists of many group elements instead of just one.

8.2 Master Theorem

To analyze the new assumption of this paper, we generalize the master theorem of Katz et al. \[24\] to use prime order bilinear groups instead of composite order bilinear groups and to use multiple groups elements in the target instead of just one element.

Let \(G, G_T\) be cyclic bilinear groups of order \(p\) where \(p\) is a large prime. The bilinear map is defined as \(e : G \times G \rightarrow G_T\). In the generic group model, a random group element of \(G, G_T\) is represented as a random variable \(P_i, R_i\) respectively where \(P_i, R_i\) are chosen uniformly in \(\mathbb{Z}_p\). We say that a random variable has degree \(t\) if the maximum degree of any variable is \(t\). Then we can naturally define the dependence and independence of random variables as in Definition 8.2.1.

**Definition 8.2.1.** Let \(P = \{P_1, \ldots, P_n\}\), \(T_0 = \{T_{0,1}, \ldots, T_{0,m}\}\), \(T_1 = \{T_{1,1}, \ldots, T_{1,m}\}\) be random variables over \(G\) where \(T_{0,i} \neq T_{1,i}\) for all \(1 \leq i \leq m\), and let \(R = \{R_1, \ldots, R_n\}\) be random variables over \(G_T\). We say that \(T_b\) is dependent on \(A\) if there exists constants \(\{\alpha_i\}, \{\beta_i\}\) such that

\[
\sum_{i=1}^{m} \alpha_i T_{b,i} = \sum_{i=1}^{n} \beta_i \cdot P_i
\]
where $\alpha_i \neq 0$ for at least one $i$. We say that $T_b$ is independent of $P$ if $T_b$ is not dependent on $P$.

Let $S_1 = \{(i, j) | e(T_{b,i}, T_{b,j}) \neq e(T_{b,i}, T_{b,j})\}$ and $S_2 = \{(i, j) | e(T_{b,i}, P_j) \neq e(T_{b,i}, P_j)\}$. We say that 

\[ \{e(T_{b,i}, T_{b,j})\}_{(i,j) \in S_1} \cup \{e(T_{b,i}, P_j)\}_{(i,j) \in S_2} \] 

is dependent on $P \cup R \cup \{e(T_{b,i}, T_{b,j})\}_{(i,j) \in S_1} \cup \{e(T_{b,i}, P_j)\}_{(i,j) \not\in S_2}$ if there exist constants \{$\alpha_{i,j}, \{\alpha'_i\}_j, \{\beta_i\}_j, \{\gamma_i\}_j, \{\delta_i\}_j$\} such that

\[
\sum_{(i,j) \in S_1} \alpha_{i,j} \cdot e(T_{b,i}, T_{b,j}) + \sum_{(i,j) \not\in S_1} \alpha'_{i,j} \cdot e(T_{b,i}, T_{b,j}) + \sum_{(i,j) \in S_2} \beta_{i,j} \cdot e(T_{b,i}, P_j) + \sum_{(i,j) \not\in S_2} \beta'_{i,j} \cdot e(T_{b,i}, P_j)
\]

\[
= \sum_i \sum_j \gamma_i \cdot e(P_i, P_j) + \sum \delta_i \cdot R_i.
\]

where $\alpha_{i,j} \neq 0$ for at least one $(i, j) \in S_1$ or $\beta_{i,j} \neq 0$ for at least one $(i, j) \in S_2$. We say that 

\[ \{e(T_{b,i}, T_{b,j})\}_{(i,j) \in S_1} \cup \{e(T_{b,i}, P_j)\}_{(i,j) \in S_2} \] 

is independent of $P \cup R \cup \{e(T_{b,i}, T_{b,j})\}_{(i,j) \in S_1} \cup \{e(T_{b,i}, P_j)\}_{(i,j) \not\in S_2}$ if $\{e(T_{b,i}, T_{b,j})\}_{(i,j) \in S_1} \cup \{e(T_{b,i}, P_j)\}_{(i,j) \not\in S_2}$ is not dependent on $P \cup R \cup \{e(T_{b,i}, T_{b,j})\}_{(i,j) \in S_1} \cup \{e(T_{b,i}, P_j)\}_{(i,j) \not\in S_2}$.

Using the above dependence and independence of random variables, we can generalize the master theorem of Katz et al. as Theorem[24].

**Theorem 8.2.1.** [24] Let $P = \{P_1, \ldots, P_a\}, T_0 = \{T_{0,0}, \ldots, T_{0,m}\}, T_1 = \{T_{1,0}, \ldots, T_{1,m}\}$ be random variables over $\mathbb{G}$ where $T_{0,i} \neq T_{1,i}$ for all $1 \leq i \leq m$, and let $R = \{R_1, \ldots, R_v\}$ be random variables over $\mathbb{G}_T$. Consider the following experiment in the generic group model:

An algorithm is given $P = \{P_1, \ldots, P_a\}$ and $R = \{R_1, \ldots, R_v\}$. A random bit $b$ is chosen, and the adversary is given $T_b = \{T_{b,0}, \ldots, T_{b,m}\}$. The algorithm outputs a bit $b'$, and succeeds if $b' = b$. The algorithm’s advantage is the absolute value of the difference between its success probability and 1/2.

Let $S_1 = \{(i, j) | e(T_{0,i}, T_{0,j}) \neq e(T_{1,i}, T_{1,j})\}$ and $S_2 = \{(i, j) | e(T_{0,i}, P_j) \neq e(T_{1,i}, P_j)\}$. If $T_b$ is independent of $P$ for all $b \in \{0, 1\}$, then any algorithm $A$ issuing at most $q$ instructions has an advantage at most $O(q^2 t/p)$.

The master theorem of Katz et al. still holds in prime order bilinear groups since the dependent equation of an adversary can be used to distinguish the target $T_b$ of the assumption. Additionally, it still holds when the target consists of multiple group elements since the adversary can only make a dependent equation in Definition[8.2.1].

### 8.3 Analysis of Our Assumptions

To prove that our assumption holds in the generic group model by applying the master theorem of previous section, we only need to show the independence of $T_0, T_1$ random variables.

#### 8.3.1 P3DH Assumption

Using the notation of previous section, the decisional P3DH assumption can be written as follows

\[
P = \{1, X, A, XA, B, XB, AB + XZ_1, Z_1, C + XZ_2, Z_2\}, \quad R = \{1\} \]

\[
T_0 = \{ABC + XZ_3, Z_3\}, \quad T_1 = \{D + XZ_3, Z_3\}.
\]
The $T_1$ has a random variable $D$ that does not exists in $P$. Thus the independence of $T_1$ is easily obtained. Therefore, we only need to consider the independence of $T_0$. First, $T_0$ is independent of $P$ since $T_0$ contains $Z_3$ that does not exist in $P$. For the independence of $\{e(T_{0,i},T_{0,j})\}_{(i,j)\in S_1} \cup \{e(T_{0,i},P_j)\}_{(i,j)\in S_2}$, we should define two sets $S_1, S_2$. We obtain that $S_1 = \{(1,1), (1,2), (2,1), (2,2)\}$. However, $e(T_{0,i},T_{0,j})$ contains $Z_3^2$ because of $Z_3$ in $T_0$, and $Z_3^2$ can not be obtained from the right part of the equation in Definition 8.2.1. Thus, the constants $\alpha_{i,j}$ should be zero for all $(i,j)$. From this, we obtain the simple equations as follows

$$\sum_{(i,j)\in S_2} \beta_{i,j} \cdot e(T_{b,i},P_j) + \sum_{(i,j)\notin S_2} \beta'_{i,j} \cdot e(T_{b,i},P_j) = \sum_{i} \sum_{j} \gamma_{i,j} \cdot e(P_i,P_j) + \sum_{i} \delta_{i} \cdot R_i.$$  

The set $S_2$ is defined as $\{(i,j) \mid \forall i,j\}$ because of $D$ in $T_1$. However, $Z_3$ in $T_0$ should be removed to construct a dependent equation since $Z_3$ does not exists in $P,R$. To remove $Z_3$ from the left part of the above simple equation, two random variables $Y, XY$ should be paired with $T_{0,i}$ for some $Y \in P$. If $Z_3$ is remove in the left part of the above simple equation, then the left part has at least a degree 3 and it contains $ABC$. To have a degree 3 in the right part of the above simple equation, $AB + XZ_1, Z_1$ should be used. However, the right part of the above equation can not contain $ABC$ since $C, XC$ do not exist in $P$. Therefore, the independence of $T_0$ is obtained.
Chapter 9

Conclusion

In this thesis, we proposed efficient HVE schemes with short tokens. We first presented the efficient HVE schemes that have the constant size of tokens and the constant cost of pairing computations in decryption. The scheme was based on composite order bilinear groups where the order is a product of three primes. Additionally, we constructed a scheme in asymmetric bilinear groups where there are no efficiently computable isomorphisms between two groups. Next, we presented a general framework that converts HVE schemes from composite order bilinear groups to prime order bilinear groups. Using this framework, we constructed HVE schemes that are secure under any kind of pairing types. Finally, we proposed a fully secure HVE scheme with short tokens in composite order bilinear groups by adapting the dual system encryption technique.

There are many interesting problems that should be solved. The first one is to construct a delegatable HVE scheme with short tokens. The delegation property was achieved in hierarchical identity based encryption and attribute based encryption. Though Shi and Waters constructed a delegatable HVE scheme, the decryption cost of their construction is proportional to the number of attributes in tokens. The second one is to construct a HVE scheme with constant size of ciphertexts. In HIBE, the scheme with constant size of ciphertexts was proposed. In HVE, it is not easy because the scheme should support wild-card in tokens. The third one is to construct a fully secure HVE scheme without any restrictions on the capability of the adversary.
Bibliography

[1] Michel Abdalla, Mihir Bellare, Dario Catalano, Eike Kiltz, Tadayoshi Kohno, Tanja Lange, John
Malone-Lee, Gregory Neven, Pascal Paillier, and Haixia Shi. Searchable encryption revisited: Consis-
tency properties, relation to anonymous ibe, and extensions. In Victor Shoup, editor, Advances in

key encryption. In Colin Boyd, editor, Advances in Cryptology - ASIACRYPT 2001, volume 2248 of


privacy. In Alessandra Cherubini, Mario Coppo, and Giuseppe Persiano, editors, Theoretical Computer

oracles. In Christian Cachin and Jan Camenisch, editors, Advances in Cryptology - EUROCRYPT

ciphertext. In Ronald Cramer, editor, Advances in Cryptology - EUROCRYPT 2005, volume 3494 of

with keyword search. In Christian Cachin and Jan Camenisch, editors, Advances in Cryptology -
EUROCRYPT 2004, volume 3027 of Lecture Notes in Computer Science, pages 506–522. Springer,
2004.

[8] Dan Boneh and Matthew K. Franklin. Identity-based encryption from the weil pairing. In Joe Kilian,
editor, Advances in Cryptology - CRYPTO 2001, volume 2139 of Lecture Notes in Computer Science,


editor, Theory of Cryptography - TCC 2005, volume 3378 of Lecture Notes in Computer Science, pages


